WIR 151

3.8, 3.9, 3.10 Solutions

3.8

1. $f(x) = \frac{1}{x}$ $f'(x) = \frac{-1}{x^2}$ $f''(x) = \frac{2}{x^3}$

$f^{(3)}(x) = \frac{-2 \cdot 3}{x^4}$ $f^{(4)}(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$

$f^{(n)}(x) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{x^{n+1}}(-1)^n$

2. $f(x) = x^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}x^{-1/2}$ $f''(x) = -\frac{1}{4}x^{-3/2}$

$f^{(3)}(x) = \frac{3}{8}x^{-5/2}$ $\boxed{f^{(4)}(x) = \frac{-15}{16}x^{-7/2}}$

3.

a) $f(x) = \tan x$ $f'(x) = \sec^2 x$ Using the Chain Rule,

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

b) $f(x) = \frac{x}{x^2+1}$ $f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$$f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2) \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

$$= (x^2+1)\left[-2x(x^2+1) - 4x(1-x^2)\right] \Big/ (x^2+1)^4$$

$$\boxed{f''(x) = \frac{2x^3 - 6x}{(x^2+1)^3}}$$

3.9

4. $x(t) = 3t^2+2 \qquad y(t) = t^4 - 32t$

$x'(t) = 6t \qquad y'(t) = 4t^3 - 32$

Horizontal tangent: Solve $y'(t)=0$, $x'(t) \neq 0$.

$4t^3 - 32 = 4(t^3-8) = 0$ if $t=2$ only.
$x'(2) \neq 0$

$x(2) = 14$ and $y(2) = -48$ $\boxed{\text{H.T. at } (14, -48)}$

Vertical tangent: Solve $x'(t)=0$, $y'(t) \neq 0$.

$t=0$ makes $x'=0$, $y' \neq 0$
$x(0) = 2 \qquad y(0) = 0$ so $\boxed{\text{V.T. at } (2, 0)}$

5. $x(t) = \sqrt{t+1} \qquad y(t) = \sqrt[3]{t-8}$

$x'(t) = \frac{1}{2}(t+1)^{-1/2}$ is never 0 so no Vertical tangents?

$y'(t) = \frac{1}{3}(t-8)^{-2/3}$ is never 0, so no Horizontal tangents?

$$\frac{y'(t)}{x'(t)} = \frac{2\sqrt{t+1}}{3\sqrt[3]{(t-8)^2}}$$ Shows

H.T. at $t=-1$ and V.T. at $t=8$

~~but~~ $(x(-1), y(-1)) = \boxed{(0, \sqrt[3]{-9}) \text{ H.T.}}$ $\boxed{(x(8), y(8)) = (3, 0) \text{ V.T.}}$

5. Alternatively: $t = x^2 - 1 \quad x \geq 0$

so $\quad y = (x^2 - 9)^{1/3}$

$$y' = \frac{2}{3} \frac{x}{(x^2-9)^{2/3}}$$

Horizonal tangents when $x = 0 \quad y = \sqrt[3]{-9}$

Vertical tangent when $x^2 = 9 \quad x \geq 0$

so $x = 3, \; y = 0$

3.10

7. We assume the snowball is a sphere and maintains the spherical shape as it melts.

$V = \frac{4}{3}\pi r^3 \quad$ with $r = 2$

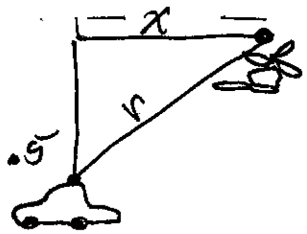
$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} \quad \text{(chain Rule)}$$

Given that $\frac{dr}{dt} = -2$ when $r = 10$,

$$\frac{dV}{dt} = \frac{4}{3}\pi(300)(-2) = -800\pi \ \text{cm}^3/\text{min}$$

at that instant.

8. Let V = speed of the car.



$$\frac{dr}{dt} = 30 \quad \text{is given}$$

$$\frac{dx}{dt} = 120 - V$$

$$r^2 = x^2 + (.5)^2 = x^2 + .25$$

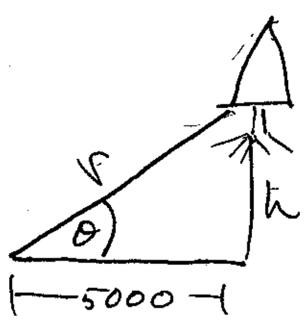
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} \qquad \text{When } r = 1, \; x = \sqrt{1 - .25}$$

$$x = \sqrt{.75}$$

$$\cancel{2}(1)\cdot 30 = \cancel{2}(\sqrt{.75})(120 - V)$$

$$V = 120 - \frac{30}{\sqrt{.75}} \approx 85.36 \text{ mph}$$

9.



$$\tan\theta = \frac{h}{5000}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{5000}\frac{dh}{dt}$$

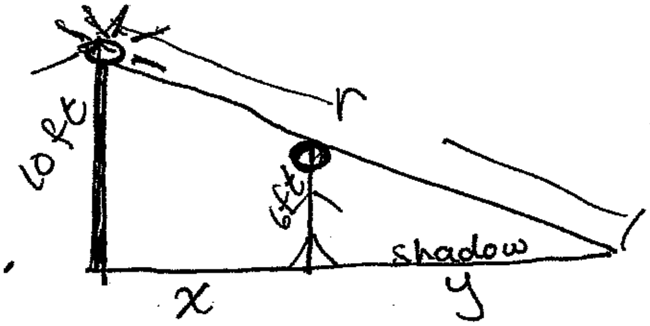
When $h = 2500$, $r = \sqrt{31.25} \times 1000$ and

$$\sec^2\theta = \frac{31.25}{25} = 1.25$$

$$1.25\frac{d\theta}{dt} = \frac{1}{5000}(400)$$

$$\frac{d\theta}{dt} = \frac{4}{50}\cdot\frac{4}{5} = \frac{16}{250}\,\frac{\text{radians}}{\text{sec}}$$

$$= .064 \text{ radians/sec}$$

10. 

$\frac{dx}{dt} = 2$ ft/sec is given

Find $\frac{dr}{dt}$: $\frac{r}{10} = \frac{y}{6}$ by similar Δ's,

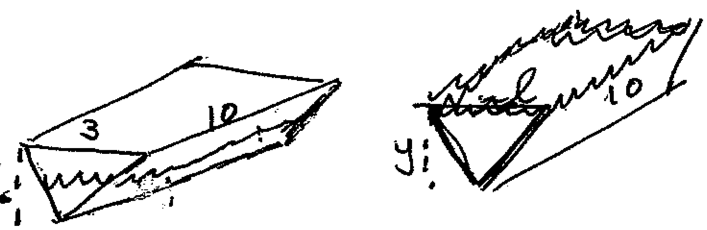
$r = \frac{5}{3} y$. $\frac{dr}{dt} = \frac{5}{3} \frac{dy}{dt}$

Find $\frac{dy}{dt}$: $\frac{x+y}{10} = \frac{y}{6}$ by similar Δ's

$\frac{1}{10} x = \left(\frac{1}{6} - \frac{1}{10}\right) y$

$\frac{1}{10} x = \frac{1}{15} y \qquad y = 1.5 x$

so $\frac{dy}{dt} = 1.5 \frac{dx}{dt}$ and $\frac{dr}{dt} = \frac{5}{3}(1.5) \frac{dx}{dt} = 5 \frac{ft}{sec}$

11. 

y = depth of the water
l = distance across the top of the water.

The volume of water is $\left(\frac{1}{2} l y\right) \times 10 = 5 l y = V$

By similar triangles, $\frac{3}{1} = \frac{l}{y}$ so $l = 3y$.

Then $V = 5ly = 15y^2$

$\frac{dV}{dt} = 30 y \frac{dy}{dt}$ When $y = 6$ in. $= \frac{1}{2}$ ft.,

$12 = 30\left(\frac{1}{2}\right) \frac{dy}{dt}$ so $\frac{dy}{dt} = \frac{12}{15} = \frac{4}{5}$ ft/min