151 Week in Review

5.7, 6.1, 6.2 solutions

1 a) $f(x) = 2x^{-3} - \frac{4}{x}$

$\int f(x)\,dx = 2\frac{x^{-2}}{-2} - 4\ln|x| + C$

$= -\frac{1}{x^2} - 4\ln|x| + C$

b) $\int f(x)\,dx = \frac{5}{7}x^{7/5} + 2\left(\frac{4}{3}\right)x^{3/4} + C$

$= \frac{5}{7}x^{7/5} + \frac{8}{3}x^{3/4} + C$

c) $\int f(x)\,dx = -\cos x - \frac{3}{4}x^{4/3} + C$

d) $\int f(x)\,dx = \sec x + C$

2. a) $F(x) = \arctan x + C$

$F(1) = \arctan 1 + C = \pi$

$= \pi/4 + C = \pi \quad C = 3\pi/4$

$F(x) = \arctan x + 3\pi/4$

b) $F(x) = \sin x + \tan x + C$

$F(\pi/4) = \frac{\sqrt{2}}{2} + 1 + C = \frac{\sqrt{2}}{2}; \; C = -1$

$F(x) = \sin x + \tan x - 1$

2c) $F''(x) = 2x^3$ so $F'(x) = \frac{1}{2}x^4 + c_1$

so $F(x) = \frac{1}{10}x^5 + c_1 x + c_2$

$F(0) = 0 + 0 + c_0 = 2$ so $c_0 = 2$

$F(1) = 4$ so $\frac{1}{10} + c_1 + 2 = 4$

$c_1 = \frac{39}{10}$

$F(x) = \frac{1}{10}x^5 + \frac{39}{10}x + 2$

d) $F''(x) = 6x^{-2}$ so $F'(x) = -6x^{-1} + c_1 = \frac{-6}{x} + c_1$

$F'(x) = -\frac{6}{x} + c_1$ so $F(x) = -6\ln|x| + c_1 x + c_0$

$F(1) = -6\ln 1 + c_1 + c_0 = 3$ so $\boxed{c_1 + c_0 = 3}$

$F(e) = -6\ln e + c_1 e + c_0 = 3$ so $\boxed{c_1 e + c_0 = 9}$

Solving: $c_1 e + c_0 = 9$ add

$-c_1 - c_0 = -3$

$(e-1)c_1 = 6$ $\quad c_1 = \frac{6}{e-1}$

$c_0 = 3 - \frac{6}{e-1}$

$F(x) = -6\ln|x| + \frac{6}{e-1}x + 3 - \frac{6}{e-1}$

2e) $F''(x) = \sin x$ so $F'(x) = -\cos x + C_1$

$F'(0) = 3$ so $-\cos 0 + C_1 = 3$ $\quad C_1 = 4$

$F'(x) = -\cos x + 4$

$F(x) = -\sin x + 4x + C_0$

$F(0) = 2 = -\sin 0 + 0 + 2$ so $C_0 = 2$

$F(x) = -\sin x + 4x + 2$

3. $\vec{a}(t) = -g\,\vec{j}$ where g is gravity

$\vec{v}(t) = -gt\,\vec{j} + \vec{v}(0)$

$= 500\cos 30^\circ\,\vec{i} + (500\sin 30^\circ - gt)\,\vec{j}$

$= 250\sqrt{3}\,\vec{i} + (250 - gt)\,\vec{j}$

a) $\vec{r}(t) = 250\sqrt{3}\,t\,\vec{i} + (250t - \frac{1}{2}gt^2)\,\vec{j} + \vec{r}(0)$

$= 250\sqrt{3}\,t\,\vec{i} + (250t - \frac{1}{2}gt^2 + 200)\,\vec{j}$

b) Max ht. occurs when the $\vec{j}$ component of $\vec{v}(t)$ is 0; $t = \frac{250}{g}$

The $\vec{j}$ comp. of $\vec{r}(t)$ at $t = \frac{250}{g}$ is

$(250)\left(\frac{250}{g}\right) - \frac{1}{2}g\left(\frac{250}{g}\right)^2 + 200 \approx 3388\text{m}$

c) Using the quadratic formula to solve $y(t)=0$, we find the projectile hits the ground at about 51.808 sec.

The magnitude of $\vec{v}(51.808)$ is about 503.9 m/s

4. $a(t) = -40$ and $v(0) = 73\frac{1}{3}$ ft/sec (1mi = 5280 ft)

$v(t) = -40t + 73\frac{1}{3}$

$d(t) = -20t^2 + 73\frac{1}{3}t$

The car stops at $t = \dfrac{73\frac{1}{3}}{40}$ sec $= \frac{11}{6}$ sec.

$d(\frac{11}{6}) \cong 67.\overline{2}$ ft

5. Let t_s be the time it takes the car to stop.
$d(t_s) = 160$ ft.

$v(t) = -40t + v_0$ so $t_s = \dfrac{v_0}{40}$

$d(t) = -20t^2 + v_0 t$

$160 = -20\left(\dfrac{v_0}{40}\right)^2 + v_0\left(\dfrac{v_0}{40}\right) = \dfrac{v_0^2}{80}$ Solving:

$v_0 = 80\sqrt{2}$ ft/sec

≈ 77.14 m.p.h

6. Since car A passes car B twice, car B must pass car A at some time between.

Let $d(t)$ be the distance between them. Show $d''(t_0)=0$ at some t_0.

Let $t_1 < t_2 < t_3$ be the 3 times at which d is 0.
By the Mean Value Thm.,
Since $d(t_1) = d(t_2) = 0$, there is some s_1 between t_1 and t_2 at which $d'(s_1)=0$.

Similarly, at some s_2 between t_2 and t_3, $d'(s_2)=0$.

Then there is some t_0 between s_1 and s_2 at which $d''(t_0)=0$.

Ch. 6

7. a) $\displaystyle\sum_{k=1}^{20}(2k+3) = 2\left[\frac{20(21)}{2}\right] + 3\cdot 20 = 480$

b) $\displaystyle\sum_{k=1}^{30}(k^2+10k+25) = \frac{30(31)(61)}{6} + 10\left(\frac{30\cdot 31}{2}\right) + 25\cdot 30$

$= 15855$

c) $\displaystyle\sum_{k=1}^{30}(k+5)^2 - \sum_{k=1}^{9}(k+5)^2$

$\displaystyle = 15855 - \left[\sum_{k=1}^{9}(k^2+10k+25)\right]$

$\displaystyle = 15855 - \left[\frac{9\cdot 10\cdot 19}{6} + 10\cdot\frac{9\cdot 10}{2} + 25\cdot 9\right] = 13895$

Another way is to reindex:

7c continued:

$$\sum_{k=15}^{35} k^2 = \sum_{k=1}^{35} k^2 - \sum_{k=1}^{14} k^2$$

$$= \frac{35 \cdot 36 \cdot 71}{6} - \frac{14 \cdot 15 \cdot 29}{6}$$

7d) $$\sum_{k=0}^{15} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{16} \left(\frac{1}{2}\right)^{k-1} = \frac{\left(\frac{1}{2}\right)^{16} - 1}{\frac{1}{2} - 1}$$

$$= 2 - \frac{1}{2^{15}} \sim 1.99997$$

7e) $$\sum_{k=1}^{10} \left(\frac{1}{3}\right)^{k-1} = \frac{\left(\frac{1}{3}\right)^{10} - 1}{\frac{1}{3} - 1} \approx 1.49997$$

8. $4^0 = 1$ $4^{\frac{1}{2}} = 2$ $4^1 = 4$ $4^{3/2} = 8$ $4^2 = 16$

$0 \quad \frac{1}{2} \quad 1 \quad 3/2 \quad 2$ $\qquad \Delta x = \frac{1}{2}$

$\frac{1}{2}[1 + 2 + 4 + 8]$ LHS

$\frac{1}{2}[2 + 4 + 8 + 16] = 15$ RHS

$$\text{Avg} = \frac{\frac{15}{2} + 15}{2} = \frac{45}{4}$$

9. $\Delta x = \frac{3}{20}$ The left endpt. of the k_{th} interval $0 \le k \le 19$ is $0 + \frac{3}{20}k$. $f\left(\frac{3}{20}k\right) = \left(\frac{3}{20}k\right)^2 = \frac{9}{400}k^2$

$$LHS = \frac{3}{20} \sum_{k=0}^{19} \frac{9}{400} k^2 \quad \text{or} \quad \frac{3}{20} \sum_{k=1}^{20} \frac{9}{400}(k-1)^2$$

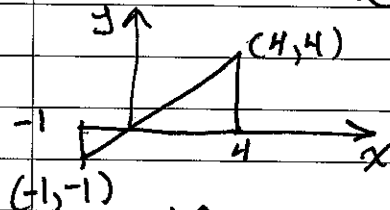
$$= \frac{3}{20} \sum_{k=1}^{19} \frac{9}{400} k^2 = \frac{27}{8000}\left(\frac{19 \cdot 20 \cdot 39}{6}\right)$$

$$= 8.33625$$

10. $\Delta x = \frac{5}{12}$

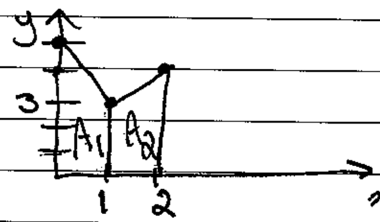
LHS: $\frac{5}{12} \sum_{k=0}^{11} \left(-1+\frac{5}{12}k\right) = \frac{163}{24}$

RHS: $\frac{5}{12} \sum_{k=1}^{12} \left(-1+\frac{5}{12}k\right) = \frac{205}{24}$



net area $= -\frac{1}{2} + \frac{1}{2}(4\times 4) = 7.5$ is the limit.

11. a)



$A = b\left(\frac{h_1+h_2}{2}\right)$

$A_1 = 1\left(\frac{3+5}{2}\right) = 4$

$A_2 = 1\left(\frac{3+4}{2}\right) = \frac{7}{2}$

$\int_0^2 f(x)\,dx = A_1 + A_2 = \frac{15}{2}$

b) $-15/2$ since $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $y=\sqrt{16-x^2}$ $\quad -4 \le x \le 4$ is the top half of a circle of radius 4.

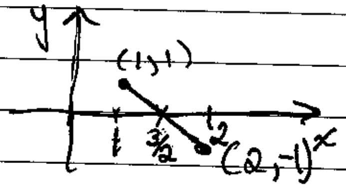
$\int_{-4}^{4} \sqrt{16-x^2}\,dx = \frac{1}{2}\left(\pi \cdot 4^2\right) = 8\pi$

d) To find $\int_0^1 x^2\,dx$ we use Riemann sums:

RHS: $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k^2}{n^2}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3}$

$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$

To find $\int_1^2 f(x)\,dx = \int_1^2 3-2x\,dx$, use geometry:

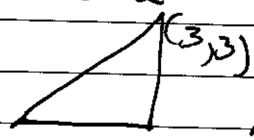


The net area is 0.

So the end result is $\frac{1}{3} = \int_0^2 f(x)\,dx$

12. The total area between f(x) and the x-axis is $\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{5}{6}$

13. a) $\int_0^3 x\,dx$



$A = \frac{1}{2}\cdot 3\cdot 3 = 9/2$

$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx = 7$

$\int_0^3 [x + f(x)]\,dx = \frac{9}{2} + 7 = \frac{23}{2}$

b) $4\int_0^3 f(x)\,dx - 5\int_0^3 g(x)\,dx = 4\times 7 - 5\times 4 = 8$