1 a) This is a difference quotient for \( f(x) = \cos x \) at \( x = \frac{\pi}{2} \).
\[
\frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}} = \frac{\cos x - 0}{x - \frac{\pi}{2}} \rightarrow -\sin \frac{\pi}{2} = -1
\]

Without using the known derivative:
\[\cos x = \sin \left(\frac{\pi}{2} - x\right)\]
\[= -\sin \left(x - \frac{\pi}{2}\right)\]

\[\lim_{x \to \frac{\pi}{2}} \frac{-\sin\left(x - \frac{\pi}{2}\right)}{x - \frac{\pi}{2}} = -\lim_{h \to 0} \frac{-\sin h}{h} = -1\]

1 b) Another difference quotient
\[
\lim_{x \to \frac{\pi}{2}} \frac{-\sin\left(x - \frac{\pi}{2}\right)}{x - \frac{\pi}{2}} = -\lim_{h \to 0} \frac{-\sin h}{h} = -1
\]

1 c) \( \sin(x-2) \to 0 \) and \( \sin t \) is negative if \( -\pi < t < 0 \).
Since \( x \to 2 \) \( \frac{1}{\sin(x-2)} \leq 0 \) and approaches \( -\infty \) in magnitude. The limit is \( -\infty \).

1 d) \( \lim_{x \to 2} \frac{x(x-2)}{\sin(x-2)} = \lim_{x \to 2} x \lim_{h \to 0} \frac{h}{\sin h} = 2 \cdot 1 = 2 \)
(since both exist)
1e) \( \lim_{x \to 0} \frac{e^{x+2} - e^2}{x} = \) the derivative of \( e^t \) at \( t=2 \) which is \( e^2 \).

1f) \( \lim_{x \to \frac{\pi}{2}} \frac{e^{\cos x} - 1}{x - \frac{\pi}{2}} = \) the derivative of \( e^{\cos t} \) at \( t=\frac{\pi}{2} \)

\[
\frac{d}{dt} (e^{\cos t}) \bigg|_{t=\frac{\pi}{2}} = e^{\cos t} (-\sin t) \bigg|_{t=\frac{\pi}{2}} = e^{0} (-1) = -1
\]

1g) \( |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases} \) since \( x \to -\alpha \), \( |x| = -x \)

\[ -\frac{x}{x+1} \to -1 \quad x \to -\alpha \]

\[ \lim_{x \to -\infty} \frac{|x|}{x+1} = 2^{-1} = \frac{1}{2} \]

2a) \( x^2 + y^2 = 1 \quad 2x + 2yy' = 0 \quad y' = -\frac{x}{y} \)

A radial line through the center and a pt. \((a, b)\) on the circle has slope \( \frac{b}{a} \).

The tangent has slope \( -\frac{a}{b} \), so they are \( \perp \).
2b) \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)
\( \frac{2x}{4} + \frac{2yy'}{9} = 0 \)
\( \frac{x}{4} + \frac{yy'}{9} = 0 \)

At \((a, b)\) the slope is \( -\frac{9a}{4b} \) and so it is not \( \perp \) to the line with slope \( \frac{b}{a} \).

\( -\frac{9}{4} \frac{a}{b} \neq -\frac{9}{b} \) since \( 9 \neq 4 \) unless \( a = 0 \) or \( b = 0 \) (only \( \perp \) for a circle or at the extremes).

3. \( x \sec y + xy = A \)

\((\sec y + x \sec y \tan y y') + (y + xy') = 0\)

\((x \sec y \tan y + x) y' = -\sec y - y\)

\( y' = -\frac{\sec y - y}{x \sec y \tan y + x} \)

4. Multiply by \( x - 2y \) to avoid the quotient rule.

\( x^3 - 2x^2y = x + 2y \)

\( 3x^2 - 4xy - 2x^2y' = 1 + 2y' \)

Substituting \( x = 1 \) and \( y = 0 \):

\( 3 - 2y' = 1 + 2y' \)

\( 2 = 4y' \)

\( \frac{1}{2} = y' \)

The tangent line is

\( y = \frac{1}{2}(x - 1) + 0 \)

\( y = \frac{1}{2}x - \frac{1}{2} \)
5. \[ x'(t) = (-4 \cos t)(-\sin t) = 4 \cos t \sin t \]

\[ y'(t) = 2 \tan t \sec^2 t \]

What value of \( t \) makes \( x(t) = 0 \) and \( y(t) = 1 \)?

\[ 1 - 2 \cos^2 t = 0 \quad t = \pm \frac{\pi}{4} \quad \tan^2 t = 1 \quad \text{also at } \pm \frac{\pi}{2} \]

At \( t = \frac{\pi}{4} \):

\[ \frac{y'(t)}{x'(t)} = \frac{2(2)}{2} = 2 \]

\[ \tan \frac{\pi}{4} = 1 \]

\[ \sec \frac{\pi}{4} = \sqrt{2} \]

\[ \sec^2 \frac{\pi}{4} = 2 \]

Tangent line is \( y = 2x + 1 \)

At \( t = -\frac{\pi}{4} \):

\[ y'(t) = -\frac{\sqrt{2}}{2} \quad \text{and} \quad x'(t) = -2 \]

so the slope and tangent are the same as at \( t = \frac{\pi}{4} \).

6. \[ 127 = 4(3) + 3 \quad f''(x) = f'''(x) \quad \text{for} \quad f(x) = \cos x \]

\[ f'(x) = -\sin x \quad f''(x) = -\cos x \quad \boxed{f'''(x) = \sin x} \]
7. \( y = Ae^{-x} + Bxe^{-x} = (A + Bx)e^{-x} \)

Using the product and chain rules:

\[ y' = Be^{-x} - (A + Bx)e^{-x} = (B - A - Bx)e^{-x} \]

\[ y'' = -Be^{-x} - (B - A - Bx)e^{-x} = (-2B + A + Bx)e^{-x} \]

\[ y'' + 2y' + y = \left[ A + Bx + 2B - 2A - 2Bx \right] e^{-x} = 0 \]

8. \( y = e^{ax} \)

\[ y' = ae^{ax} \]

\[ y'' = a^2e^{ax} \]

\[ y'' - y' - y = 0 \text{ if } (a^2 - a - 1)e^{ax} = 0 \]

Since \( e^{ax} \) is never 0 we must have \( a^2 - a - 1 = 0 \).

\[ a = \frac{-1 \pm \sqrt{5}}{2}, \quad \frac{1 + \sqrt{5}}{2}, \quad \frac{1 - \sqrt{5}}{2} \]

9. \( f'(x) = p'(x)e^x + p(x)e^x = (p'(x) + p(x))e^x \)

\[ f'(x) = \left( p''(x) + p'(x) \right)e^x + \left( p'(x) + p(x) \right)e^x \]

\[ = \left[ p''(x) + 2p'(x) + p(x) \right] e^x \]

\[ f''(x) = \left[ p'''(x) + 3p''(x) + 3p'(x) \right] e^x + \left[ p'(x) + 2p'(x) + p(x) \right] e^x \]

\[ = \left[ p'''(x) + 3p''(x) + 3p'(x) + p(x) \right] e^x \]
Volume of water

\[
V = \frac{1}{3} \pi r^2 y = \frac{1}{3} \pi \left( \frac{1}{16} y^2 \right) y = \frac{\pi}{48} y^3
\]

\[
V = \frac{\pi}{48} y^3 \text{ so } \frac{dV}{dt} = \frac{3\pi}{48} y^2 \frac{dy}{dt} = \frac{\pi}{16} y^2 \frac{dy}{dt}
\]

When \( y = 5 \):

\[
2 = \frac{\pi}{16} (25) \frac{dy}{dt}
\]

\[
\frac{32}{2} = \frac{dy}{dt} \text{ in ft/min}
\]

11. \( x = \) distance from car A to the intersection B

\[
r = \text{distance from A to B}
\]

Find \( \frac{dr}{dt} \) when \( x = 300 \), \( y = 400 \)

\[
r^2 = x^2 + y^2
\]

\[
2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]

When \( x = 300 \) and \( y = 400 \), \( r = 500 \).

\[
500 \frac{dr}{dt} = 300(40) + 400(30)
\]

(The conversion factor cancels out)

\[
5 \frac{dr}{dt} = -24 \quad \frac{dr}{dt} = -\frac{24}{5} \approx -4.8 \text{ mph}
\]

\( r \) is decreasing by 4.8 mph.
12. \( h(x) = f(g(x)) \) Find \( h \) and \( h' \) at \( x = 1 \).

The tangent line to \( f \) tells us \( f(2) \) and \( f'(2) \)
The tangent line to \( g \) tells us \( g(1) \) and \( g'(1) \)

Since \( g(1) = 2 \), we can find:
\[
h(1) = f(g(1)) = f(2) = 4(2) + 5 = 13
\]
\[
h'(x) = f'(g(x))g'(x) \quad \text{the chain rule}
\]
\[
h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 4(5) = 20
\]
\[
y = 20(x-1) + 13 = 20x - 7 \quad \text{is the tangent line to } h \text{ at } x = 1.
\]

13. Find the tangent line to \( f(x) = \sqrt{x} \)
at \( x = 32 \) since we know \( \frac{5}{\sqrt{32}} \) and 32 is "close" to 30,
\[
f'(x) = \frac{1}{2}x^{-4/5} = \frac{1}{\sqrt[5]{5^4}} = \frac{1}{5(2^4)} = \frac{1}{80}
\]
\[
L(x) = \frac{1}{80}(x-32) + 2 \quad \text{The linear approx. to } \frac{5}{\sqrt{30}} \text{ is}
\]
\[
L(30) = \frac{1}{80}(30-32) + 2
\]
\[
= 2 - \frac{2}{80} = 1.975
\]
Use Newton's Method to find a root of $x^5 - 30$ using $x_1 = 2$ as the first guess.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^5 - 30}{5(2^4)} = 2 - \frac{32}{80} = 1.975$$

15. $f(x) = \frac{x+1}{x-2}$
   - Domain $(-\infty, 2) \cup (2, \infty)$ or $x \neq 2$
   - Range $(-\infty, 1) \cup (1, \infty)$ or $y \neq 1$

   H.A. $y = 1$   V.A. $x = 2$

   find $f^{-1}$: $x = \frac{y+1}{y-2}$

   $$xy - 2y = y + 1$$
   $$xy - y = 2x + 1$$
   $$(x-1)y = 2x + 1$$
   $$y = \frac{2x + 1}{x - 1}$$

   Domain $x \neq 1$
   Range $y \neq 2$

   H.A. $y = 2$   V.A. $x = 1$

16. $y = e^x \quad \rightarrow \quad f^{-1}: x = e^y$

   Using implicit differentiation:

   $$1 = e^y y'$$
   $$\frac{1}{e^y} = y'$$
   but $e^y = x$ so

   $$\frac{1}{x} = y'$$
17. \[ f(-3x+5) = 2 \quad x=1 \]

\[ f^{-1}(2) = 1 \]

\[ (f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{-3} \]

\[ f'(1) = -3 \text{ since that is the slope of the tangent at 1.} \]

The tangent to \( f^{-1} \) is \( y = -\frac{1}{3}(x-2) + 1 \)

\[ = -\frac{1}{3}x + \frac{5}{3} \]

(and this is the inverse to the line \( y = -3x+5 \))