151 Exam 2 Review Solutions

1 a) This is a difference quotient for $f(x) = \cos x$ at $x = \pi/2$.

$$\frac{f(x) - f(\pi/2)}{x - \pi/2} = \frac{\cos x - 0}{x - \pi/2} \longrightarrow -\sin \tfrac{\pi}{2} = -1$$

Without using the known derivative:

$$\cos x = \sin(\tfrac{\pi}{2} - x)$$

$\cos x = \sin(\pi/2 - x)$

$$= -\sin(x - \tfrac{\pi}{2})$$

$$\lim_{x \to \pi/2} \frac{-\sin(x - \pi/2)}{x - \pi/2} = -\lim_{h \to 0} \frac{\sin h}{h} = -1$$

1 b) Another difference quotient

$$\frac{\sin(x + \pi/6) - \sin \pi/6}{x} \xrightarrow[x \to 0]{} \cos \tfrac{\pi}{6} = \frac{\sqrt{3}}{2}$$

1 c) $\sin(x-2) \xrightarrow[x \to 2]{} 0$ and $\sin t$ is negative if $-\pi < t < 0$.

Since $x \to 2^-$ $\frac{1}{\sin(x-2)} < 0$ and approaches ∞ in magnitude. The limit is $\boxed{-\infty}$.

1 d) $$\lim_{x \to 2} \frac{x(x-2)}{\sin(x-2)} = \lim_{x \to 2} x \lim_{h \to 0} \frac{h}{\sin h} = 2(1) = 2$$

(since both exist)

1e) $\lim_{x\to 0} \frac{e^{x+2} - e^{2}}{x}$ = the derivative of e^{t} at $t=2$ which is e^2.

1f) $\lim_{x\to \pi/2} \frac{e^{\cos x} - 1}{x - \pi/2}$ = the derivative of $e^{\cos t}$ at $t = \pi/2$

$$\frac{d}{dt}\left(e^{\cos t}\right)\Big|_{t=\pi/2} = e^{\cos t}(-\sin t)\Big|_{t=\pi/2} = e^{0}(-1) = -1$$

1g) $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$ Since $x \to -\infty$, $|x| = -x$

$$\frac{-x}{x+1} \underset{x\to -\infty}{\longrightarrow} -1$$

$$\lim_{x\to -\infty} 2^{\frac{|x|}{x+1}} = 2^{-1} = \tfrac{1}{2}$$

2a) $x^2 + y^2 = 1$ $2x + 2yy' = 0$ $y' = -\frac{x}{y}$

A radial line through the center and a pt. (a, b) on the circle has slope $\frac{b}{a}$.

The tangent has slope $-\frac{a}{b}$, so they are $\perp$.

2b) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ $\qquad \frac{2x}{4} + \frac{2yy'}{9} = 0$

$$\frac{x}{4} + \frac{yy'}{9} = 0$$

$$y' = -\frac{9x}{4y}$$

At (a,b) the slope is $-\frac{9}{4}\frac{a}{b}$ and so it is not $\perp$ to the line with slope $\frac{b}{a}$.

$-\frac{9}{4}\frac{a}{b} \neq -\frac{a}{b}$ since $9 \neq 4$ (only $\perp$ for a circle or at the extremes)

unless $a = 0$ or $b = 0$

3. $x \sec y + xy = A$

$$(\sec y + x \sec y \tan y\, y') + (y + xy') = 0$$

$$(x \sec y \tan y + x)\, y' = -\sec y - y$$

$$y' = \frac{-\sec y - y}{x \sec y \tan y + x}$$

4. Multiply by $x - 2y$ to avoid the quotient rule.

$$x^3 - 2x^2 y = x + 2y$$

$$3x^2 - 4xy - 2x^2 y' = 1 + 2y'$$

Substituting $x = 1$ and $y = 0$:

$$3 - 2y' = 1 + 2y'$$

$$2 = 4y'$$

$$\frac{1}{2} = y'$$

→ The tangent line is

$$y = \frac{1}{2}(x - 1) + 0$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

5. $x'(t) = (-4\cos t)(-\sin t) = 4\cos t \sin t$

$y'(t) = 2\tan t \sec^2 t$

What value of t makes $x(t) = 0$ and $y(t) = 1$?

$-\frac{\pi}{2} < t < \frac{\pi}{2}$

$1 - 2\cos^2 t = 0 \quad t = \pm\pi/4 \quad \tan^2 t = 1$ also at $\pm\pi/4$.

At $t = \pi/4$: $\frac{y'(t)}{x'(t)} = \frac{2(2)}{2} = 2$

$\tan\frac{\pi}{4} = 1$
$\sec\frac{\pi}{4} = \sqrt{2}$
$\sec^2\frac{\pi}{4} = 2$

Tangent line is $y = 2x + 1$

At $t = -\pi/4$: $y'(t) = -4$ and $x'(t) = -2$
so the slope and tangent are the same as at $t = \pi/4$.

6. $127 = 4(31) + 3 \quad f^{(127)}(x) = f'''(x)$ for $f(x) = \cos x$.

$f'(x) = -\sin x \quad f''(x) = -\cos x \quad \boxed{f'''(x) = \sin x}$

7. $y = Ae^{-x} + Bxe^{-x} = (A+Bx)e^{-x}$

Using the product and chain rules:

$$y' = Be^{-x} - (A+Bx)e^{-x} = (B - A - Bx)e^{-x}$$

$$y'' = -Be^{-x} - (B-A-Bx)e^{-x} = (-2B + A + Bx)e^{-x}$$

$$y'' + 2y' + y = [\underset{y}{A+Bx} + \underset{2y'}{2B - 2A - 2Bx} \underset{y''}{- 2B + A + Bx}]e^{-x}$$

$$= 0$$

8. $y = e^{ax} \quad y' = ae^{ax} \quad y'' = a^2 e^{ax}$

$y'' - y' - y = 0$ if $(a^2 - a - 1)e^{ax} = 0$

Since e^{ax} is never 0 we must have

$a^2 - a - 1 = 0 \quad a = \frac{+1 \pm \sqrt{5}}{2} \qquad \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

9. $f'(x) = p'(x)e^x + p(x)e^x = (p'(x) + p(x))e^x$

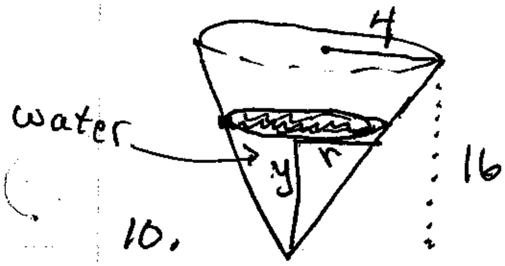
$$f''(x) = (p''(x) + p'(x))e^x + (p'(x) + p(x))e^x$$

$$= [p''(x) + 2p'(x) + p(x)]e^x$$

$$f'''(x) = [p'''(x) + 2p''(x) + p'(x)]e^x + [p''(x) + 2p'(x) + p(x)]e^x$$

$$= [p'''(x) + 3p''(x) + 3p'(x) + p(x)]e^x$$

10.



$\frac{dV}{dt} = 2$ Find $\frac{dy}{dt}$ when $y = 5$.

$\frac{r}{y} = \frac{4}{16}$ so $r = \frac{1}{4} y$

Volume of water $= V = \frac{1}{3}\pi r^2 y = \frac{1}{3}\cdot\pi\left(\frac{1}{16}y^2\right) y = \frac{\pi}{48} y^3$

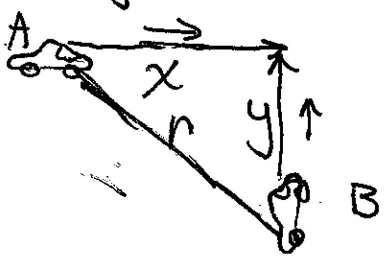
$V = \frac{\pi}{48} y^3$ so $\frac{dV}{dt} = \frac{3\pi}{48} y^2 \frac{dy}{dt} = \frac{\pi}{16} y^2 \frac{dy}{dt}$

When $y = 5$: $2 = \frac{\pi}{16}(25)\frac{dy}{dt}$

$\frac{32}{2\pi} = \frac{dy}{dt}$ in ft/min

11. x = distance from car A to the intersection

y = " " B "



r = distance from A to B.

Find $\frac{dr}{dt}$ when $x = 300$ $y = 400$

$r^2 = x^2 + y^2$ $\quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

When $x = 300$ and $y = 400$, $r = 500$.

$500\frac{dr}{dt} = 300(-40) + 400(-30)$

(The conversion factor cancels out)

$5\frac{dr}{dt} = -24$ $\quad \frac{dr}{dt} = \frac{-24}{5} = -4.8$ mph

r is decreasing by 4.8 mph

12. $h(x) = f(g(x))$ Find h and h' at $x=1$.

The tangent line to f tells us $f(2)$ and $f'(2)$
The tangent line to g tells us $g(1)$ and $g'(1)$

Since $g(1) = 2$, we can find:

$$h(1) = f(g(1)) = f(2) = \overset{L2(2)}{4(2)+5} = 13$$

$$h'(x) = f'(g(x))\, g'(x) \quad \text{the chain rule}$$
$$h'(1) = f'(g(1))\, g'(1) = f'(2)\, g'(1) = 4\,(5) = 20$$

$y = 20(x-1) + 13 = 20x - 7$ is the tangent line to h at $x=1$.

13. Find the tangent line to $f(x) = \sqrt[5]{x}$ at $x = 32$ since we know $\sqrt[5]{32}$ and 32 is "close" to 30.

$$f'(x) = \frac{1}{5}x^{-4/5} = \frac{1}{5\left(\sqrt[5]{x}\right)^4} = \frac{1}{5(2^4)} = \frac{1}{80}$$

$L(x) = \frac{1}{80}(x - 32) + 2$ The linear approx. to $\sqrt[5]{30}$ is $L(30) = \frac{1}{80}(30 - 32) + 2$

$$= 2 - \frac{2}{80} = 1.975$$

Use
14. Newton's Method to find a root of $x^5 - 30$ using $x_1 = 2$ as the first guess.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^5 - 30}{5(2^4)} = 2 - \frac{2}{80} = 1.975$$

15. $f(x) = \frac{x+1}{x-2}$ Domain $(-\infty, 2) \cup (2, \infty)$ or $x \neq 2$
Range $(-\infty, 1) \cup (1, \infty)$ or $y \neq 1$

H.A. $y = 1$ V.A. $x = 2$

find f^{-1}: $x = \frac{y+1}{y-2}$

$$x(y-2) = y+1$$
$$xy - y = 2x + 1$$
$$(x-1)y = 2x+1$$
$$y = \frac{2x+1}{x-1}$$

Domain $x \neq 1$
Range $y \neq 2$

H.A. $y = 2$ V.A. $x = 1$

16. $y \overset{f}{=} e^x \longmapsto f^{-1}: x = e^y$ Using implicit differentiation:

$$1 = e^y y'$$
$$\frac{1}{e^y} = y'$$ but $e^y = x$ so
$$\frac{1}{x} = y'$$

17. $1 \overset{f}{\longrightarrow} -3x+5\Big|_{x=1} = 2$, $\quad 1 \overset{f^{-1}}{\longleftarrow} 2$

$f^{-1}(2) = 1$

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{-3}$$

$f'(1) = -3$ since that is the slope of the tangent at 1.

The tangent to f^{-1} is $y = -\frac{1}{3}(x-2) + 1$

$$= -\frac{1}{3}x + \frac{5}{3}$$

(and this is the inverse to the line $y = -3x+5$)