There eleven questions. Show your work. Problems 1-8 are 6 points each. Problems 9-11 are 18 points each making a total of 102 points. Partial credit will be given according to work shown.

‘An Aggie does not lie, cheat or steal or tolerate those who do.’

Good Luck!
Problems 1, 2 and 3, evaluate each integral.

1. $\int \sin^2(x) \, dx$

   Method 1
   
   $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$
   
   $\int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin(2x) + C$

   Method 2
   
   $u = \sin x \quad dv = \sin x \, dx$
   
   $du = \cos x \, dx \quad v = -\cos x$

   $-\sin x \cos x - \int \cos^2 x \, dx = -\sin x \cos x + \int \sin^2 x \, dx$

   $2\int \sin^2 x \, dx = -\sin x \cos x + x + C$

2. $\int x^2 e^{2x} \, dx$

   I by P

   $u = x^2 \quad dv = e^{2x} \, dx$

   $du = 2x \, dx \quad v = \frac{1}{2} e^{2x}$

   $\frac{1}{2} x^2 e^{2x} - \int x e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - \left( \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right)$

   $u = x \quad dv = e^{2x} \, dx$

   $du = dx \quad v = \frac{1}{2} e^{2x}$

   $= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C$
3. \[ \int \sin^3(x) \cos^2(x) \, dx = \int \sin^2(x) \cos^2(x) \sin(x) \, dx \]

\[ u = \cos(x) \]
\[ du = -\sin(x) \, dx \]

\[ = \int (1 - \cos^2(x)) \cos^2(x) \sin(x) \, dx \]
\[ = \int (-u^2) u^2 (-du) \]
\[ = \int (u^2 - u^4) \, du \]
\[ = -\frac{1}{3} \cos^3(x) + \frac{1}{5} \cos^5(x) + C \]

4. Find the area of the region bounded by the graphs of \( x = y^2 \) and \( y = x - 2 \).

\[ x = y^2 \quad \text{and} \quad x = y + 2 \]

Find y-limits:
\[ y^2 - y - 2 = 0 \quad (y - 2)(y + 1) = 0 \]

\[ \int_{-1}^{2} \left[ y + 2 - y^2 \right] \, dy \]
\[ = \left[ \frac{1}{2} y^2 + 2y - \frac{1}{3} y^3 \right]_{-1}^{2} \]
\[ = 2 + 4 - \frac{8}{3} - (\frac{1}{2} - 2 + \frac{1}{3}) \]
\[ = \frac{9}{2} \]
5. The force to hold a spring \( \frac{1}{5} \) m beyond its natural length is 2N. Find the work done in stretching the spring from its natural length to \( \frac{3}{5} \) m beyond its natural length.

\[
\text{Force} = kx = \frac{1}{5} k = 2 \quad \Rightarrow \quad k = 10
\]

\[
\text{Work} = \int_{0}^{\frac{3}{5}} 10 x \, dx = 5 x^2 \bigg|_{0}^{\frac{3}{5}}
\]

\[
= 5 \left( \frac{9}{25} \right) = \frac{9}{5} \text{ Joules}
\]

6. The base of a solid is the region bounded by \( y = x^2 \) and \( y = 2 \). Cross sections perpendicular to the y-axis are squares. Find the volume of the solid.

[Diagram of a solid with a cross section at \( y \) showing \( x = \sqrt{y} \), \( 2x(y) = \text{side of square at } y \), and \( A(y) = 4y \).]

\[
\int_{0}^{2} 4y \, dy = 2y^2 \bigg|_{0}^{2} = 8
\]
7. A 70 pound, 35 foot rope of uniform density hangs freely from the top of a tall building. Find the work done in pulling 10 feet of rope to the top of the building.

\[ \text{Total weight} = 70 \text{ lbs} \]
\[ \rho g = \frac{70 \text{ lb}}{35 \text{ ft}} = 2 \text{ lb/ft} \]

\[ \int_0^{10} (70 - 2y) \, dy = 700 - 100 = 600 \text{ ft-lbs} \]

8. Use area to find the average value on the interval \([0,6]\) of \(f(x)\) shown in the graph.

\[ \frac{1}{6} \int_0^6 f(x) \, dx = \frac{1}{6} \left( 12 \right) = 2 \]
(18 points)

9. Evaluate each integral.

a) \[ \int \frac{\ln x}{\sqrt{x}} \, dx \]
   \[ u = \ln x, \quad du = \frac{1}{x} \, dx \]
   \[ dv = \frac{1}{\sqrt{x}} \, dx = x^{-\frac{1}{2}} \, dx \]
   \[ v = 2x^{\frac{1}{2}} \]

\[ 2x^{\frac{1}{2}} \ln x - \int 2x^{\frac{1}{2}} \frac{1}{x} \, dx = 2x^{\frac{1}{2}} \ln x - 2\int x^{-\frac{1}{2}} \, dx \]

\[ = 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} + C \]

b) \[ \int \sec^4(x) \sin^2(x) \, dx \]
   \[ \sec^4(x) \sin^2(x) = \tan^2(x) \sec^2(x) \]
   \[ \int \sec^2 x \tan^2 x \, dx = \frac{1}{3} \tan^3 x + C \]

\[ \sec^2 \sin^2 x = \frac{\sin^2 x}{\cos^4 x} = \frac{1}{\cos^2 x} \left( \frac{\sin x}{\cos^2 x} \right) = \sec^2 x \tan x \]

\[ \int \sec^5(x) \tan^3(x) \, dx = \int \sec^4 x \tan^2 x (\sec x \tan x) \, dx \]
   \[ u = \sec x, \quad du = \sec x \tan x \, dx \]
   \[ \tan^2 x = \sec^2 x - 1 \]

\[ \int u^5 - u^4 \, du = \frac{1}{6} \sec^6 x - \frac{1}{5} \sec^5 x + C \]
(18 points)

10. a) Find the volume of the solid formed when the region bounded by 
\( y = 2x, \ y = -2x, \ \text{and} \ x = 1 \) is revolved about the line \( y = 2 \).

\[
\text{Using washers:} \\
R = 2 - (-2x) = 2 + 2x \\
r = 2 - 2x \\
\pi \int_0^1 (2+2x)^2 - (2-2x)^2 \, dx = \pi \int_0^1 16x \, dx = \pi \left[ 8x^2 \right]_0^1 = 8\pi
\]

Note: If you use shells you cannot use symmetry. The region in the lower portion is further from the rotation axis.

b) The region bounded by \( y = \sqrt{x^2 + 5}, \ y = x, \ x = 0, \) and \( x = 2 \) is revolved about the \( y \)-axis. Find the volume of the resulting solid.

\[
\text{Use shells.} \quad \text{Radius} = x, \quad \text{Height} = \sqrt{x^2 + 5} - x \\
2\pi \int_0^2 \left[ x \sqrt{x^2 + 5} - x^2 \right] \, dx
\]

Subst. \( u = x^2 + 5 \) in the first integral.

\[
\int_0^2 x \sqrt{x^2 + 5} \, dx = \int_0^5 \frac{1}{2} \sqrt{u} \, du = \frac{1}{3} (u^{3/2})_0^5 = \frac{1}{3} (27 - 5^{3/2}) = \frac{2\pi}{3} (27 - 5^{3/2})
\]
11. A tank has the shape of an isosceles trapezoid as shown. The base is 4 m wide, the top is 6 m wide. The tank is 4 m high and 10 m long. Find the work done in each case. \( x \) and \( y \) are in meters. Use \( 9.8 \) for the weight density of water. You must show your coordinate system.

b) Assume the tank is full. Set up but do not evaluate the integral to find the work done in pumping all the water out the top of the tank.

Find the width of the rectangle \( y \) m up:

slope = \( 4 \)

\[ y = 4(x - 2) \]

\[ x = \frac{y}{4} + 2 = \frac{y + 8}{4} \]

\[ w = 2x = \frac{y + 8}{2} \]

\[ dV = 5(y + 8)\,dy \]

The distance from \( y \) to the top is \( H - y \).

\[ W = \int_{0}^{H} 5g(4.5 - y)(y + 8)\,dy \]

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c) Set up but do not evaluate the work to pump the water out a spout 0.5 m above the top of the tank until the depth of the water is 1 m.

This changes the distances the wts. are lifted to \( 4.5 - y \) and we pump out \( y \)-levels 1 to \( H \).

\[ 5g\int_{1}^{H} (4.5 - y)(y + 8)\,dy \]