There eleven questions. Show your work. Problems 1-8 are 6 points each. Problems 9-11 are 18 points each making a total of 102 points. Partial credit will be given according to work shown.

‘An Aggie does not lie, cheat or steal or tolerate those who do.’

Good Luck!
Problems 1, 2 and 3, evaluate each integral.

1. $\int x^2e^{2x} \, dx$
   \[ u = x^2 \quad dv = e^{2x} \, dx \]
   \[ du = 2x \, dx \quad v = \frac{1}{2} e^{2x} \]
   \[ \frac{1}{2} x^2 e^{2x} - \int xe^{2x} \, dx \]
   \[ u = x \quad dv = e^{2x} \, dx \]
   \[ du = dx \quad v = \frac{1}{2} e^{2x} \]
   \[ = \frac{1}{2} x^2 e^{2x} - \left[ \frac{1}{2} xe^{2x} - \int \frac{1}{2} e^{2x} \, dx \right] \]
   \[ = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} xe^{2x} + \frac{1}{4} e^{2x} + C \]

2. $\int \cos^2(x) \, dx = \int \frac{1}{2} + \frac{1}{2} \cos(2x) \, dx$
   \[ = \frac{1}{2} x + \frac{1}{4} \sin(2x) + C \]
3. \[ \int \cos^3(x) \sin^2(x) \, dx = \int \cos x \sin^2 x (\cos x \, dx) \quad u = \sin x \quad du = \cos x \, dx \]

\[ = \int (1 - \sin^2 x) \sin^2 x \, dx \quad u^2 = u^4 \, du = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C \]

4. Find the area of the region bounded by the graphs of \( x = y^2 \) and \( 2y = x - 3 \).

\[ \int_{-1}^{3} (2y + 3 - y^2) \, dy \]

\[ = y^2 + 3y - \frac{1}{3} y^3 \bigg|_{-1}^{3} \]

\[ = 9 + 9 - 9 - \left( \frac{1}{3} \cdot 9 \right) \]

\[ = 11 - \frac{27}{3} = \frac{12}{3} \]
5. The force to hold a spring \( \frac{1}{4} \) m beyond its natural length is 3N. Find the work done in stretching the spring from its natural length to \( \frac{3}{4} \) m beyond its natural length.

\[
F = kx \quad 3N = \frac{1}{4}k \quad k = 12
\]

\[
\int_{0}^{\frac{3}{4}} 12x \, dx = 6x^2 \bigg|_{0}^{\frac{3}{4}} = 6\left(\frac{9}{16}\right) = \frac{27}{8}
\]

6. The base of a solid is the region bounded by \( y = x^2 \) and \( y = 3 \).
Cross sections perpendicular to the y-axis are squares. Find the volume of the solid.

The side of the square at \( y \) is \( 2x \) so the area is \( 4x^2 = 4y \)

\[
\int_{0}^{3} 4y \, dy = 2y^2 \bigg|_{0}^{3} = 18
\]
7. A 120 pound, 30 foot rope of uniform density hangs freely from the top of a tall building. Find the work done in pulling 12 feet of rope to the top of the building.

\[ \text{Total Weight} = 120 \quad \rho g = 4 \text{ lbs/ft} \]

\[ \int_{0}^{12} (120 - 4y) \, dy = \left(20y - 2y^2\right) \left|^{12}_{0} \right. \]

\[ = 1440 - 288 = 1152 \text{ ft-lbs} \]

8. Use area to find the average value on the interval \([0,6]\) of \(f(x)\) shown in the graph.

\[ \frac{1}{6} \int_{0}^{6} f(x) \, dx = \frac{1}{6} \cdot 12 = 2 \]
(18 points)

9. Evaluate each integral.

a) \[ \int \sec^5(x) \sin^3(x) \, dx \]

\[ \frac{2}{3} \sec^3 x \sin^3 x = \sec^3 x \tan x \]

\[ = \int \tan^3 x \sec^3 x \, dx = \frac{1}{4} \tan^4 x + C \]

b) \[ \int \frac{\ln x}{x^{1/3}} \, dx \]

\[ u = \ln x \]
\[ du = \frac{1}{x} \, dx \]
\[ dv = x^{-\frac{1}{3}} \, dx \]
\[ v = \frac{2}{3} x^{2/3} \]

\[ \frac{3}{2} (\ln x) x^{2/3} - \frac{3}{2} \int \frac{1}{x} x^{2/3} \, dx = \frac{3}{2} x^{2/3} \ln x - \frac{3}{2} \int x^{-\frac{1}{3}} \, dx \]

\[ = \frac{3}{2} x^{2/3} \ln x - \frac{9}{4} x^{2/3} + C \]

\[ = \frac{3}{2} x^{2/3} \ln x - \frac{9}{4} x^{2/3} + C \]

\[ \int \sec^3(x) \tan^3(x) \, dx = \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx \]

\[ = \int \sec^2 x \, dx - \int \sec^3 x \, dx \]

\[ = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C \]

u = \sec x

\[ du = \sec x \tan x \, dx \]
(18 points)

10. a) Find the volume of the solid formed when the region bounded by 
\( y = 3x, \quad y = -3x, \quad \text{and} \quad x = 1 \) is revolved about the line \( y = 3 \).

\[
\begin{align*}
R & = 3 - (-3x) \\
& = 3 - 3x \\
& \ni \int_0^1 \pi \left( (3+3x)^2 - (3-3x)^2 \right) \, dx \\
= & \pi \int_0^1 36x \, dx \\
= & 18 \pi
\end{align*}
\]

b) The region bounded by \( y = \sqrt{x^2 + 5}, \ y = x, \ x = 0, \ \text{and} \ x = 2 \) is revolved about the \( y \)-axis. Find the volume of the resulting solid.

\[
\begin{align*}
& \begin{align*}
\text{cyl. shells} & \quad r = x \\
& \quad h = \sqrt{x^2 + 5} - x \\
& \quad 2 \pi \int_0^2 x \left( \sqrt{x^2 + 5} - x \right) \, dx \\
= & 2 \pi \int_0^2 x \sqrt{x^2 + 5} \, dx \bigg|_0^2 - \int_0^2 x^2 \, dx \\
= & 2 \pi \left[ \frac{1}{2} (x^2 + 5)^{3/2} \bigg|_0^2 - \frac{1}{3} x^3 \bigg|_0^2 \right] \\
= & \frac{2 \pi}{3} \left[ 27 - 5 - \frac{8}{3} \right]
\end{align*}
\end{align*}
\]
(18 points)

11. A tank has the shape of an isosceles trapezoid as shown. The base is 6m wide, the top is 10 m wide. The tank is 4 m high and 12m long. Find the work done in each case. \( x \) and \( y \) are in meters. Use \( \rho g \) for the weight density of water. **You must show your coordinate system.**

\[ \text{The face of the tank} \]

b) Assume the tank is full. Set up but do not evaluate the integral to find the work done in pumping all the water out the top of the tank.

Cross sections at level \( y \) are rectangles of length 12 and width \( 2x \), for the face shown in the graph. \( y = 2(x-3) \)

\[ A(y) = 12(y+6) \text{ is lifted } 4-y \text{ m} \]

\[ 12\rho g \int_0^4 (y+6)(4-y) \, dy \]

c) Set up but do not evaluate the work to pump the water out a spout 0.5 m above the top of the tank until the depth of the water is 1 m.

Now the distance is \( 4.5-y \) and we only pump out from \( y=1 \) to \( y=4 \).

\[ \rho g \int_1^4 (y+6)(4.5-y) \, dy \]