There are 7 multiple choice problems worth 5 points each.
There are 4 work out problems worth 16 points each.

This makes 99 points. The other point is above if your name is readable!

"On my honor as an Aggie I have neither given nor received aid on this academic work."

Sign here __________________
Multiple Choice Section. There are 7 problems worth 5 points each in this section.

1. Evaluate \( \int_{1}^{e^2} \frac{(\ln x)^3}{x} \, dx \).
   
   a) \( \frac{8}{e^4} \)  
   b) \( \frac{e^8 - 1}{4} \)  
   c) 2  
   d) 4  
   e) 8

   Substitute \( u = \ln x \)  
   \( du = \frac{1}{x} \, dx \)  
   \( u(1) = \ln 1 = 0 \)  
   \( u(e^2) = \ln (e^2) = 2 \)

   \[
   \int_{1}^{e^2} \frac{(\ln x)^3}{x} \, dx = \int_{0}^{2} u^3 \, du = \frac{1}{4} u^4 \bigg|_{0}^{2} = 4
   \]

2. The function, \( f(x) \), is continuous on \([0, 4]\).
   The average value of \( f(x) \) on the interval \([0, 2]\) is 3. The average value of \( f(x) \) on the interval \([0, 4]\) is 5. Find the average value of \( f(x) \) on the interval \([2, 4]\).
   
   a) 2  
   b) There is not enough information.  
   c) 4  
   d) 8  
   e) 7

   \[
   \frac{1}{2} \int_{0}^{2} f(x) \, dx = 3 \quad \text{so} \quad \int_{0}^{2} f(x) \, dx = 6
   \]

   \[
   \frac{1}{4} \int_{0}^{4} f(x) \, dx = 5 \quad \text{so} \quad \int_{0}^{4} f(x) \, dx = 20
   \]

   \[
   \int_{2}^{4} f(x) \, dx = \int_{0}^{4} f(x) \, dx - \int_{0}^{2} f(x) \, dx = 20 - 6 = 14
   \]

   \[
   \frac{1}{2} \int_{2}^{4} f(x) \, dx = 7
   \]
7. The base of a solid is the region bounded by \( y = x^2 \) and \( y = 1 \). Cross-sections perpendicular to the \( y \)-axis are squares. Find the volume of the solid. Assume \( x \) and \( y \) are in the same scale and all units are appropriate.

\[ a) \ \frac{1}{2} \quad b) \ \frac{15}{15} \quad c) \ \frac{2}{3} \quad d) \ 1 \quad e) \ 2 \]

The side of the square at a certain \( y \) is \( 2x(y) \).

\[ 2x(y) = 2\sqrt{y} \]

\[ A(y) = (2\sqrt{y})^2 = 4y, \quad 0 \leq y \leq 1 \]

\[ V = \int_0^1 4y \, dy = 2y^2 \bigg|_0^1 = 2 \]
3. Find the area of the region(s) bounded by the graphs of 
\[ y = 3x^2, \quad y = 3x, \quad x = 0, \text{ and } x = 2. \]

\begin{align*}
A_1 &= \int_{0}^{1} (3x^2 - 3x) \, dx \\
&= \left. \frac{3}{2}x^2 - x^3 \right|_{0}^{1} \\
&= \frac{3}{2} - 1 = \frac{1}{2}
\end{align*}

\begin{align*}
A_2 &= \int_{1}^{2} (3x^2 - 3x) \, dx \\
&= \left. x^3 - \frac{3}{2}x^2 \right|_{1}^{2} \\
&= 8 - \frac{3}{2}(4) - (1 - \frac{3}{2}) \\
&= 8 - 6 + \frac{1}{2} = 2\frac{1}{2}
\end{align*}

\[ A_1 + A_2 = \frac{1}{2} + 2\frac{1}{2} = 3 \]

4. The region bounded by \( y = e^x, \quad y = x, \quad x = 0 \) and \( x = 1 \) is revolved about the x-axis. Find the volume of the resulting solid.

\begin{align*}
V &= \pi \int_{0}^{1} (R^2 - r^2) \, dx \\
&= \pi \int_{0}^{1} (e^x)^2 - x^2 \, dx \\
&= \pi \int_{0}^{1} e^{2x} - x^2 \, dx \\
&= \pi \left( \frac{1}{2} e^{2x} - \frac{1}{3} x^3 \right|_{0}^{1} \\
&= \pi \left( \frac{1}{2} e^{2} - \frac{1}{3} - \frac{1}{2} \right) \\
&= \pi \left( \frac{1}{2} e^{2} - \frac{5}{6} \right)
\end{align*}
5. Evaluate \( \int_{1}^{e} (3x^2 + 1) \ln x \, dx \).

\[ \text{Integrate by parts:} \]
\[ u = \ln x \quad dv = (3x^2 + 1) \, dx \quad uv - \int vdu \]
\[ du = \frac{1}{x} \, dx \quad v = x^3 + x \]

\[ (x^3 + x) \ln x - \int (x^3 + x) \cdot \frac{1}{x} \, dx = (x^3 + x) \ln x - \int x^2 + 1 \, dx \]
\[ = (x^3 + x) \ln x - \frac{1}{3} x^3 - x \bigg|_{1}^{e} = e^3 + e - \frac{1}{3} e^3 - e \]
\[ = \frac{2}{3} e^3 + \frac{2}{3} = \frac{2e^3 + 4}{3} \]

6. A 50-pound weight hangs freely from the top of a building by 20 feet of rope. The rope has uniform density of 2 pounds/foot. Find the work done in pulling the rope and weight 10 feet up, assuming the rope is rested on the top of the building as it is pulled up.

\[ \text{Total wt.} = 50 + 40 = 90 \text{ lbs} \]
\[ \int_{0}^{10} 90 - 2y \, dy = 90 \int_{0}^{10} \, dy = 800 \text{ ft-lbs} \]
Find the intersection points:

\[ x\sqrt{2+x^2} = x^3 \]

\[ x = 0 \quad \text{or} \quad \sqrt{2+x^2} = x^2 \]

\[ 2 + x^2 = x^4 \]

\[ 0 = x^4 - x^2 - 2 \]

\[ 0 = (x^2 - 2)(x^2 + 2) \]

\[ x = -\sqrt{2} \quad \sqrt{2} \]

\[ A = \left| \int_{-\sqrt{2}}^{0} x\sqrt{2+x^2} - x^3 \, dx \right| + \left| \int_{0}^{\sqrt{2}} x\sqrt{2+x^2} - x^3 \, dx \right| \]

by symmetry

\[ = 2 \left| \int_{0}^{\sqrt{2}} x\sqrt{2+x^2} - x^3 \, dx \right| \]

Let \( u = 2 + x^2 \)

\( du = 2x \, dx \)

\[ = \int_{0}^{\sqrt{2}} \frac{1}{a} \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \]

\[ = \frac{1}{3} (2+x^2)^{\frac{3}{2}} \]

\[ A = 2 \left[ \frac{1}{3} (2+x^2)^{\frac{3}{2}} - \frac{1}{4} x^4 \right] \bigg|_{0}^{\sqrt{2}} \]

\[ = 2 \left[ \frac{5}{3} - \frac{2}{3} \sqrt{2} \right] \quad \text{or} \quad \frac{10-4\sqrt{2}}{3} \]
9. 16 points
A tank in the shape of a right circular cylinder is lying horizontally as shown. The radius of the face of the tank is 2 m and the length is 10 m. The spout is 1 m above the top of the tank. The tank is full of water. Find the work done in pumping all the water out the spout. Use geometry and symmetry to evaluate the integral.

To describe the face of the tank as \( x^2 + y^2 = 4 \), the center must be at the origin. The cross sections perpendicular to the y-axis are rectangles of length 10 m, width \( 2x(y) = 2\sqrt{4-y^2} \).

\[
A(y) = 20\sqrt{4-y^2}
\]

\[
\Delta \text{Weight} = \rho g A(y) \Delta y
\]

\[
= 20\rho g \sqrt{4-y^2} \Delta y
\]

This increment of weight is lifted from level \( y \) to the spout at \( y=3 \) so \( 3-y \) m.

\[
P(y) = 3-y
\]

\[
W = \int_{-2}^{2} A(y) P(y) \, dy = 20\rho g \int_{-2}^{2} (3-y) \sqrt{4-y^2} \, dy
\]

\[
\int_{-2}^{2} \sqrt{4-y^2} \, dy = \frac{4\pi}{2} = 2\pi
\]

\[
\int_{-2}^{2} y \sqrt{4-y^2} \, dy = 0
\]

so \( W = 60\rho g (2\pi) = 120\pi \rho g \) Joules.
10. Find the volume of each solid of revolution.

7 points

a) The region bounded by \( y = 12x^2, \ y = 0 \) and \( x = 1 \) is revolved about the line \( x = -1 \).

\[ \text{Cyl. Shells: } h(x) = 12x^2 \]
\[ r(x) = x - (-1) = x + 1 \]
\[ V = 2\pi \int_0^1 (x+1)(12x^2) \, dx \]
\[ = 2\pi \int_0^1 12x^3 + 12x^2 \, dx = 2\pi \int_0^1 3x^4 + 4x^2 \, dx \]
\[ = 14\pi \]

9 points

b) The region bounded by \( y = 0, \ x = 0 \), and \( y = \cos x \) is revolved about the line \( y = 2 \).

\[ \text{Washers: } \]
\[ R = 2 - 0 = 2 \]
\[ r = 2 - \cos x \]
\[ \pi \int_0^{\frac{\pi}{2}} (2 - \cos x)^2 - (2 - \cos x) \, dx \]
\[ = \pi \int_0^{\frac{\pi}{2}} 4\cos^2 x + \cos^2 x \, dx \]
\[ = \pi \int_0^{\frac{\pi}{2}} 4\sin^2 x + \frac{1}{2} x + \frac{1}{4} \sin 2x \, dx \]
\[ = \pi \left[ 4\sin x + \frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \]
\[ = \pi \left( 4 + \frac{\pi}{4} + 0 - 0 \right) = \pi \left( 4 + \frac{\pi}{4} \right) \]
11. Find each antiderivative.
4 points
a) \( \int x \cos(2x) \, dx \)  \hspace{1cm} \text{Int. by parts}

\[
\begin{align*}
  u &= x \\
  dv &= \cos(2x) \, dx \\
  du &= dx \\
  v &= \frac{1}{2} \sin(2x)
\end{align*}
\]

\[
uv - \int vdu = \frac{1}{2} x \sin(2x) - \int \frac{1}{2} \sin(2x) \, dx
\]

\[
= \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) + C
\]

4 points
b) \( \int \tan^2(x) \sec(x) \, dx \)

\[
\begin{align*}
  u &= \sec(x) \\
  dv &= \tan(x) \sec(x) \, dx \\
  du &= \tan(x) \sec(x) \, dx \\
  v &= \sec^2(x) - 1
\end{align*}
\]

\[
\int \tan^2(x) (\tan(x) \sec(x) \, dx) =
\]

\[
= \int (\sec^2(x) - 1) (\tan(x) \sec(x) \, dx)
\]

\[
= \int (u^2 - 1) \, du = \frac{1}{3} u^3 - u + C
\]

\[
= \frac{1}{3} \sec^3(x) - \sec(x) + C
\]
4 points
c) \( \int \cos^3(x) \sin^2(x) \, dx \)

\( \cos^3 x \) has an odd exponent

so let \( u = \sin x \)
\( du = \cos x \, dx \)
\( \cos^2 x = 1 - \sin^2 x \)

\[ \int \cos^2 x \sin^2 x \, (\cos x \, dx) = \int (1 - \sin^2 x) \sin^2 x \, (\cos x \, dx) \]

\[ = \int (1-u^2)u^2 \, du = \int u^2 - u^4 \, du \]

\[ = \frac{1}{3} u^3 - \frac{1}{5} u^5 + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C \]

4 points
d) \( \int \sec^4(x) \, dx \)

let \( u = \tan x \)
\( du = \sec^2 x \, dx \)

The extra \( \sec^2 x \) is \( \tan^2 x + 1 \)

\[ \int \sec^2 x \, (\sec^2 x \, dx) = \int (\tan^2 x + 1) \, (\sec^2 x \, dx) \]

\[ = \int u^2 + 1 \, du = \frac{1}{3} u^3 + u + C \]

\[ = \frac{1}{3} \tan^3 x + \tan x + C \]