152 WIR Section 10.1 solutions

1a) $a_0 = 1$ $a_2 = \sqrt{6+1} = \sqrt{7} < 3$ $a_3 = \sqrt{6+\sqrt{7}} < \sqrt{9} = 3$

Use Induction to show $0 \le a_n \le 3$ for all n.

$n=0$ $a_0 = 1 < 3$ True

If $a_n \le 3$ then $a_{n+1} = \sqrt{6+a_n} \le \sqrt{6+3} = 3$

Show $a_n \le a_{n+1}$ for all n:

This means $a_n \le \sqrt{6+a_n}$

$$a_n^2 \le 6 + a_n$$
$$a_n^2 - a_n - 6 \le 0$$
$$x^2 - x - 6 \le 0$$ between its roots

which are -2 and 3. $(x^2-x-6) = (x-3)(x+2)$

Since $0 \le a_n \le 3$, $a_n^2 - a_n - 6 \le 0$.

Since the sequence is monotone and bounded, it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6+a_n}$$

so $L = \sqrt{6+L}$

$$L^2 - L - 6 = 0$$

L must be -2 or 3.

Since $0 \le a_n$, a_n cannot converge to -2

so $L = 3$.

1b) $a_0 = 2 \quad a_{n+1} = 4 - \frac{3}{a_n}$

If $L = \lim_{n\to\infty} a_n$ exists, it must be that

$$L = 4 - \frac{3}{L} \quad \text{so } L^2 - 4L + 3 = 0$$

$(L-3)(L-1) = 0 \quad L$ is either 1 or 3.

$a_0 = 2 \quad a_1 = 4 - \frac{3}{2} = 2.5 \quad a_2 = 4 - \frac{3}{2.5} = \frac{14}{5}$

Show $L = 3$. Show a_n is increasing and bounded.

1] Show $1 \le a_n \le 3$ for all n.

$n=0 \quad a_0 = 2$ True.

2] If $1 \le a_n \le 3$ then $4 - \frac{3}{a_n} \ge 4 - 3 = 1$

so $1 \le a_{n+1}$.

And $4 - \frac{3}{a_n} \le 4 - 1 = 3$ so $a_{n+1} \le 3$.

Show $a_{n+1} \ge a_n$ so $\{a_n\}$ is increasing.

Show $4 - \frac{3}{a_n} \ge a_n$ ie $4a_n - 3 \ge a_n^2$

$0 \ge a_n^2 - 4a_n + 3$

$x^2 - 4x + 3 \le 0$ between its roots of 1 and 3 and $1 \le a_n \le 3$ so $a_n^2 - 4a_n + 3 \le 0$.

Since a_n is bdd. and increasing and $1 \le a_n \le 3$, L exists and must equal

From the recursive definition:

1c) $a_0 = 4 \quad a_1 = 2.5 \quad a_2 = 3.4 \quad a_3 = \frac{47}{17} \approx 2.76$

It appears that $\{a_{2n}\}$ is decreasing and $\{a_{2n+1}\}$ is increasing.

If L exists it must be that $L = 1 + \frac{6}{L}$ or $L^2 - L - 6 = (L-3)(L+2) = 0$ so L is -2 or 3. We guess L is 3.

Let $b_n = a_{2n}$.

$$b_{n+1} = 1 + \frac{6}{1 + \frac{6}{b_n}} = \frac{7b_n + 6}{b_n + 6} \qquad b_0 = 4$$

1] Show $3 \le b_n \le 4$ by induction.

a) $n = 0 \quad b_0 = 4$ is True.

b) If $3 \le b_n \le 4$ then $\frac{7b_n + 6}{b_n + 6} \le \frac{7(4)+6}{3+6} \le 4$

and $3 \le \frac{7b_n + 6}{b_n + 6}$ iff $3b_n + 18 \le 7b_n + 6$
iff $12 \le 4b_n$
iff $3 \le b_n$ ✓
which is true.

2] Show $\{b_n\}$ is decreasing.

$b_{n+1} \le b_n$ if $\frac{7b_n + 6}{b_n + 6} \le b_n$

which means $0 \le b_n^2 - b_n - 6$

which is true since $3 \le b_n \le 4$.

($x^2 - x - 6$ is pos. there)

1c) Show $\{a_{2n+1}\} = \{c_n\}$ is increasing to 3.

1] $0 \le c_n \le 3$ for all n:

a) $c_0 = a_1 = 2.5$ so at $n=0$ it is true.

b) If $0 \le c_n \le 3$ then $c_{n+1} \le \dfrac{7c_n+6}{c_n+6}$ is also since

$$\frac{7c_n+6}{c_n+6} \le 3 \text{ iff } c_n^2 - c_n - 6 \ge 0$$

which it is since $c_n \in [0,3]$.

$x^2 - x - 6 > 0$ on $(0,3)$

also $7c_n + 6$ and $c_n + 6$ are positive so $c_{n+1} \ge 0$.

2] $c_{n+1} \ge c_n$ iff $\dfrac{7c_n+6}{c_n+6} \ge c_n$

Again, this is true in the interval $[0,3]$.

Since $\{a_{2n+1}\}$ is increasing & bdd, it must converge and can't converge to -2 so $L=3$.

Also $\{a_{2n}\}$ is decreasing and bdd. so it also converges to 3.

2. a) $\frac{n}{n^2+4} = \frac{1}{n+4/n} \xrightarrow[n\to\infty]{} 0$ so $\frac{(-1)^n n}{n^2+4} \xrightarrow[n\to\infty]{} 0$

b) $\left\{\frac{n^2}{3^n}\right\}$ Consider $\frac{x^2}{3^x}$ and use L'Hospital's rule.

$$\frac{x^2}{3^x} \sim \frac{2x}{3^x \ln 3} \sim \frac{2}{3^x (\ln 3)^2} \xrightarrow[x\to\infty]{} 0$$

so $\frac{n^2}{3^n} \xrightarrow[n\to\infty]{} 0$

c) $\frac{2^n+4n}{5^n} = \frac{2^n}{5^n} + \frac{4n}{5^n} = \left(\frac{2}{5}\right)^n + \frac{4n}{5^n}$

Both terms approach 0 as $n \to \infty$ so the limit is 0.

d) $\lim\limits_{n\to\infty} (1+3/n)^n = \lim\limits_{n\to\infty} \left[\left(1+\frac{1}{n/3}\right)^{n/3}\right]^3 = e^3$

e) $\frac{(n!)^2}{(2n)!} = \frac{n}{2n} \cdot \frac{n-1}{2n-1} \cdot \ldots \cdot \frac{1}{n+1} \cdot \frac{\cancel{n!}}{\cancel{n!}}$

$< 1 \cdot 1 \cdot 1 \ldots \cdot \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$

$0 \le \frac{(n!)^2}{(2n)!} < \frac{1}{n+1}$ so the limit is 0.

2f) $\left(1+\frac{2}{n}\right)^{\sqrt{n}} = \left[\left(1+\frac{2}{n}\right)^{n}\right]^{\frac{1}{\sqrt{n}}}$ Looks like $(e^2)^{1/\sqrt{n}} \to 1$

Apply L'Hospital's Rule to $\ln a_n$.

$$\ln\left(1+\frac{2}{n}\right)^{\sqrt{n}} = \sqrt{n}\,\ln\left(1+\frac{2}{n}\right)$$

$$= \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{\sqrt{n}}} = \frac{\ln(n+2)-\ln n}{1/\sqrt{n}}$$

$$\sim \frac{\frac{1}{n+2}-\frac{1}{n}}{-\frac{1}{2n^{3/2}}} =$$

$$= -2n^{3/2}\left(\frac{-2}{n(n+2)}\right)$$

$$= \frac{4\sqrt{n}}{n+2} = \frac{4}{\sqrt{n}+2/\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

$\ln a_n \to 0 \ (n \to \infty)$

so $a_n \to e^0 = 1$ $(n \to \infty)$.

2g) $\ln a_n = \frac{1}{n}\ln n = \frac{\ln n}{n} \xrightarrow[n\to\infty]{} 0$

so $a_n \to e^0 = 1 \ (n \to \infty)$

3. $|(-1)^n \sin n| \le 1$ so $|a_n| \le \frac{1}{n^p} \xrightarrow[n\to\infty]{} 0$

since $p > 0$

so $\lim_{n\to\infty} a_n = 0$

4. $\dfrac{n(n-1)(n-2)(n-3)\cdot\cdots\cdot 2\cdot 1}{n\cdot n\cdot n\cdot n\cdot\cdots\cdot n\cdot n}$

$= \underbrace{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdot\cdots\cdot\frac{2}{n}}_{\text{all are } <1}\cdot\frac{1}{n}$

$< \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

so $0 \le a_n \le \frac{1}{n}$ so $\lim_{n\to\infty} a_n = 0$

5. $\lim_{n\to\infty} \arctan\frac{n}{\sqrt{n^2+3n}} = \arctan\left(\lim_{n\to\infty}\frac{n}{\sqrt{n^2+3n}}\right)$

since $\arctan x$ is continuous.

$\frac{n}{\sqrt{n^2+3n}} = \frac{1}{\sqrt{1+3/n}} \xrightarrow[n\to\infty]{} 1$ $\arctan 1 = \pi/4$

6. $a_{n+1} = \sqrt{2+a_n}$ $a_0 = \sqrt{2}$

<u>If</u> the limit L exists, it must be a root of $L = \sqrt{2+L}$, $L^2 - L - 2 = 0$

$(L-2)(L+1) = 0$

Since all a_n's are positive, $L = 2$.

1] Show $0 \leq a_n \leq 2$ for all n.

a) For $n=0$, $0 \leq a_0 = \sqrt{2} \leq 2$ is true.

b) If $0 \leq a_n \leq 2$ then

$$a_{n+1} \leq \sqrt{2+2} = 2,$$

2] Show $\{a_n\}$ is increasing.

$a_{n+1} = \sqrt{2+a_n} \geq a_n$ if $0 \geq a_n^2 - a_n - 2$

This is because all a_n's are between the roots of $x^2 - x - 2$.

So $\lim_{n\to\infty} a_n$ exists and must be 2.