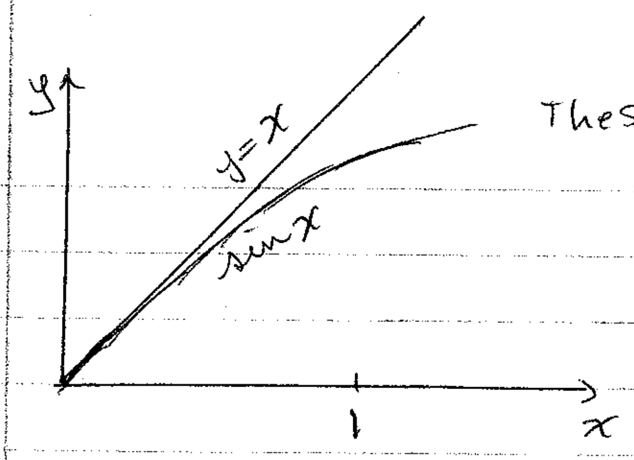
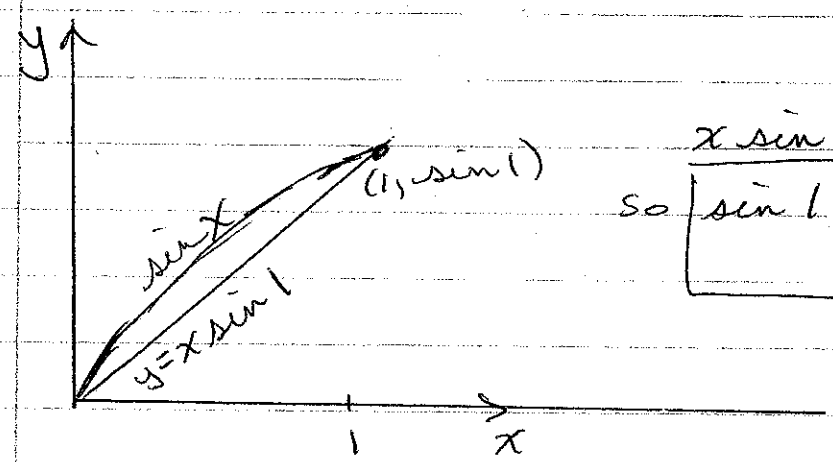
Math 152 WIR Solutions 8.9, 9.3, 9.4



These inequalities are used in #1.

$0 \le \sin x \le x$ on $[0,1]$

so $\boxed{0 \le \dfrac{\sin x}{x} \le 1 \text{ on } [0,1].}$



$x \sin 1 \le \sin x$ on $[0,1]$

so $\boxed{\sin 1 \le \dfrac{\sin x}{x} \text{ on } [0,1].}$

1 a) $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ converges since $0 \le \dfrac{\sin x}{x} \le 1$

b) $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$ converges since

$$0 \le \frac{\sin x}{x^{3/2}} = \frac{\sin x}{x} \cdot \frac{1}{x^{1/2}} \le \frac{1}{x^{1/2}} \text{ and}$$

$$\int_0^1 \frac{1}{x^{1/2}}\,dx = \lim_{t \to 0} 2\sqrt{x}\,\Big|_t^1 = 2 - \lim_{t \to 0} 2\sqrt{t} = 2$$

1c) $\int_0^1 \frac{\sin x}{x^2}\,dx$ diverges since

$$\frac{\sin x}{x^2} = \frac{\sin x}{x}\cdot\frac{1}{x} \geq \frac{\sin 1}{x} \text{ and}$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0} \ln x\Big|_t^1 = \lim_{t\to 0}\left[-\ln t\right] = +\infty.$$

1d) $\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges since

$|\cos x| \leq 1$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty} \frac{-1}{x}\Big|_1^T = \lim_{T\to\infty}\left(\frac{-1}{T}+1\right) = 1.$$

1e) $\int_1^\infty \frac{x}{(x^3+1)^{1/2}}\,dx$ diverges by comparison to

$\int \frac{1}{\sqrt{x}}\,dx$:

$$x^3+1 \leq 2x^3 \quad \text{for } x \geq 1$$

$$(x^3+1)^{1/2} \leq \sqrt{2}\,x^{3/2}$$

$$\frac{x}{(x^3+1)^{1/2}} \geq \frac{x}{\sqrt{2}\,x^{3/2}} = \frac{1}{\sqrt{2}\,x^{1/2}}$$

Since $\int_1^\infty \frac{1}{\sqrt{2}x^{1/2}}\,dx = \lim_{T\to\infty} \frac{1}{\sqrt{2}}\cdot 2\sqrt{x}\Big|_1^T = \infty$,

the integral given also diverges.

1f) $\int_1^\infty \frac{x}{(x^5+1)^{1/2}}\,dx$ converges by comparison to $\frac{x}{x^{5/2}} = \frac{1}{x^{3/2}}$,

$$\frac{x}{(x^5+1)^{1/2}} \le \frac{x}{x^{5/2}} = \frac{1}{x^{3/2}}$$ and

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{T\to\infty} -2x^{-1/2}\Big|_1^T = \lim_{T\to\infty}\left(\frac{-2}{\sqrt{T}}+2\right) = 2$$

converges.

2 a) & b) Find $\int \frac{\ln x}{x^{3/2}}\,dx$ by parts:

$u = \ln x \qquad dv = x^{-3/2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -2x^{-1/2}$

$$-2\frac{\ln x}{\sqrt{x}} + \int 2x^{-3/2}\,dx$$

$$= -\frac{2\ln x}{\sqrt{x}} - \frac{4}{\sqrt{x}} + C$$

2a) $\lim_{T\to 0^+}\left[-\frac{2\ln x}{\sqrt{x}} - \frac{4}{\sqrt{x}}\Big|_T^1\right] = \lim_{T\to 0^+}\left[-4 + \frac{2\ln T}{\sqrt{T}} + \frac{4}{\sqrt{T}}\right]$

$= +\infty$ diverges

2b) $\lim_{T\to\infty}\left[\frac{-2\ln x}{\sqrt{x}} - \frac{4}{\sqrt{x}} \Big|_1^T \right]$

$= \lim_{T\to\infty}\left[\frac{-2\ln T}{\sqrt{T}} - \frac{4}{\sqrt{T}} + 4 \right] = 4$

2c) & d)

Find $\int x e^{-x}\,dx$ by parts:

$u = x \qquad dv = e^{-x}dx$
$du = dx \qquad v = -e^{-x}$

$\int x e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$

c) $\lim_{T\to\infty}\left[-xe^{-x} - e^{-x} \Big|_0^T \right] = \lim_{T\to\infty}\left[-Te^{-T} - e^{-T} + 1 \right]$

$= 1$

d) $\lim_{T\to -\infty}\left[-xe^{-x} - e^{-x} \Big|_T^0 \right] = \left[-1 - \lim_{T\to-\infty}(-Te^{-T} - e^{-T}) \right.$

$= +\infty$ diverges

e) $\int_{-1}^{1} \frac{1}{x}\, dx$ only exists if

$\int_{-1}^{0} \frac{1}{x}\, dx$ and $\int_{0}^{1} \frac{1}{x}\, dx$ both exist.

$$\int_0^1 \frac{1}{x}\, dx = \lim_{T \to 0^+} \ln x \Big|_T^1 = +\infty \text{ diverges.}$$

Note: $\ln|x| \Big|_{-1}^{1} = 0 - 0 = 0$ but $\frac{1}{x}$

not continuous in $(-1, 1)$ so FTC does not apply.

f) $$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx = \lim_{T \to 0} \int_T^1 \frac{\ln x}{\sqrt{x}}\, dx$$

Integrate by parts:

$u = \ln x$ $\qquad dv = \frac{1}{\sqrt{x}}\, dx$

$du = \frac{1}{x}\, dx$

$\qquad v = 2\sqrt{x}$

$$2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}\, dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

$\lim_{T \to 0^+} \left[2\sqrt{x}\ln x - 4\sqrt{x} \Big|_T^1 \right] = -4$ Use L'Hospital's Rule on $\frac{\ln T}{1/\sqrt{T}}$ to show

$\sqrt{T}\ln T \to 0$

as $T \to 0$

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9.3

Arc length

3a) $x(t) = t^2 \sin t \quad y(t) = t^2 \cos t \quad 0 \le t \le \pi$

$x'(t) = 2t \sin t + t^2 \cos t$

$y'(t) = 2t \cos t - t^2 \sin t$

$[x'(t)]^2 = 4t^2 \sin^2 t + 2t^3 \sin t \cos t + t^4 \cos^2 t$

$[y'(t)]^2 = 4t^2 \cos^2 t - 2t^3 \sin t \cos t + t^4 \sin^2 t$

$[x'(t)]^2 + [y'(t)]^2 = 4t^2 + t^4 = t^2[4 + t^2]$

$dL = \sqrt{t^2(4+t^2)}\,dt = \left(t\sqrt{4+t^2}\right)dt \quad t \ge 0$

$L = \int_0^{\pi} t\sqrt{4+t^2}\,dt = \frac{1}{2}\int_4^{4+\pi^2} u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_4^{4+\pi^2}$

Substitute $u = 4 + t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$= \boxed{\frac{1}{3}(4+\pi^2)^{3/2} - \frac{8}{3}}$

3 b) $x'(t) = \sin t + t\cos t \qquad 0 \le t \le \pi$

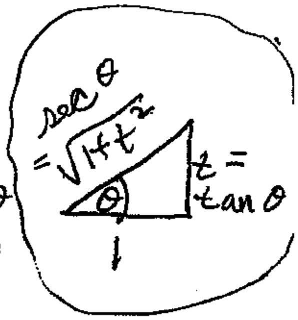
$y'(t) = \cos t - t\sin t$

$(x')^2 + (y')^2 = 1 + t^2$

$dL = \sqrt{1+t^2}\, dt$

$L = \int_0^{\pi} \sqrt{1+t^2}\, dt$ Substitute $t = \tan\theta$

$dt = \sec^2\theta\, d\theta$



$I = \int \sec^3\theta\, d\theta$ is done by parts:

$u = \sec\theta \qquad dV = \sec^2\theta\, d\theta$

$du = \sec\theta\tan\theta\, d\theta \qquad V = \tan\theta$

$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$= \sec\theta\tan\theta - \int(\sec^3\theta - \sec\theta)\, d\theta$

$2I = \sec\theta\tan\theta + \int \sec\theta\, d\theta =$

$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$

$I = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$

Substituting back:

$L = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left|\sqrt{1+t^2} + t\right| \Big|_0^{\pi}$

$L = \frac{1}{2}\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln\left(\sqrt{1+\pi^2} + \pi\right)$

4. $y = \ln(\cos x) \quad -\pi/6 \le x \le \pi/3$

$y' = \dfrac{-\sin x}{\cos x} = -\tan x$

$dL = \sqrt{1+\tan^2 x}\; dx = \sec x \; dx$

$L = \displaystyle\int_{-\pi/6}^{\pi/3} \sec x \; dx = \ln(\sec x + \tan x)\Big|_{-\pi/6}^{\pi/3}$

$= \ln(2+\sqrt{3}) - \ln\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = \ln(2+\sqrt{3}) - \ln\left(\frac{1}{\sqrt{3}}\right)$

$= \ln(2+\sqrt{3}) + \ln\sqrt{3}$

$= \ln(2\sqrt{3}+3)$

$= \boxed{\ln(2\sqrt{3}+3)}$

5. $y = x^{2/3} \quad 0 \le x \le 8$

Method I] $x = y^{3/2} \quad 0 \le y \le 4 \qquad x' = \frac{3}{2} y^{1/2}$

$\displaystyle\int_0^4 dL = \int_0^4 \sqrt{1 + \frac{9}{4}y}\; dy \qquad (x')^2 = \frac{9}{4} y$

$= \frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}y\right)^{3/2}\Big|_0^4 = \frac{8}{27}\left[\left(1+\frac{72}{4}\right)^{3/2} - 1\right]$

$= \frac{8}{27}\left(19^{3/2} - 1\right)$

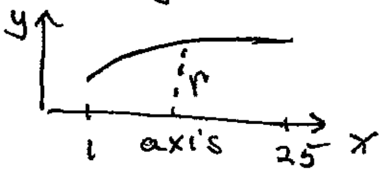
Method II] $y' = \frac{2}{3} x^{-1/3} \qquad dL = \sqrt{1 + \frac{4}{9}x^{-2/3}}\; dx$

$= x^{-1/3}\sqrt{x^{2/3} + \frac{4}{9}}\; dx$

Substitute $u = x^{2/3} + 4/9$

9.4 Surface Area $\quad SA = \int 2\pi r \, dL$

6. a) $y = \sqrt{x} \quad 1 \le x \le 25$ is rotated about the x-axis



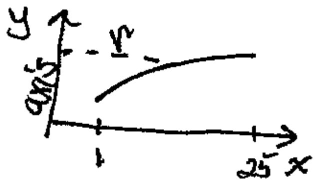
$r = y = \sqrt{x}$

$y' = \frac{1}{2\sqrt{x}} \quad dL = \sqrt{1 + \frac{1}{4x}} \, dx$

$$SA = \int_1^{25} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = 2\pi \int_1 \sqrt{x + \tfrac{1}{4}} \, dx$$

$$= 2\pi \left(\frac{2}{3}\right)\left(x + \tfrac{1}{4}\right)^{3/2} \Big|_1^{25} = \frac{4\pi}{3}\left[25.25^{3/2} - 1.25^{3/2}\right]$$

b) $y = \sqrt{x} \quad 1 \le x \le 25$ is rotated about the y-axis.



$r = x \quad dL = \sqrt{1 + \frac{1}{4x}} \, dx$

$$SA = 2\pi \int_1^{25} x \sqrt{1 + \frac{1}{4x}} \, dx$$

$$= 2\pi \int \sqrt{x^2 + \tfrac{1}{4}x} \, dx$$

which can be done with $x + \frac{1}{8} = \frac{1}{8} \sec\theta$

7. $y = \cos x \quad 0 \le x \le \frac{\pi}{2}$ rotated about the x-axis.

$$r = y = \cos x \qquad dL = \sqrt{1+\sin^2 x}\; dx$$

$$SA = 2\pi \int_0^{\pi/2} \cos x \sqrt{1+\sin^2 x}\; dx$$

$$= 2\pi \int_0^1 \sqrt{1+u^2}\; du \quad \text{where } u = \sin x$$

$\int \sqrt{1+u^2}\,du$ is done in 16
$u = \tan\theta$

$$= 2\pi \left[\frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2}\ln\left(\sqrt{1+u^2} + u\right) \right]_0^1$$

$$= 2\pi \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right) \right]$$

8a) $y = e^x \quad 0 \le x \le 1$ rotated about the x-axis.

$$r = y = e^x \qquad dL = \sqrt{1+e} \; dx$$

$$2\pi \int_0^1 e^x \sqrt{1+(e^x)^2}\; dx$$

$(e^x)^2 = e^{2x}$ but we leave it as $(e^x)^2$ for the subst.

$$u = e^x$$

$$= 2\pi \int_1^e \sqrt{1+u^2}\; du$$

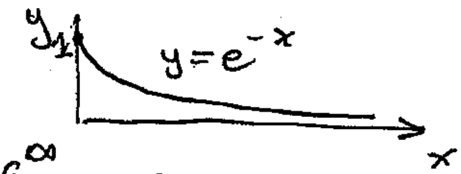
$$= 2\pi \left[\frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2}\ln\left(\sqrt{1+u^2} + u\right) \right] \Bigg|_1^e$$

$$= 2\pi \left[\frac{1}{2} e\sqrt{1+e^2} + \frac{1}{2}\ln\left(\sqrt{1+e^2} + e\right) - \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right) \right]$$

8.b. $y = e^x \quad 0 \le x \le 1$ rotated about the y-axis

$r = x \qquad dL = \sqrt{1 + (e^x)^2}\, dx = \sqrt{1 + e^{2x}}\, dx$

$SA = 2\pi \int_0^1 x\sqrt{1+e^{2x}}\, dx$ I think we need a computer for this one.

9. 

rotated about the x-axis.

$2\pi \int_0^\infty e^{-x}\sqrt{1+(e^{-x})^2}\, dx \qquad r = e^{-x} \qquad dL = \sqrt{1+(e^{-x})^2}\, d$

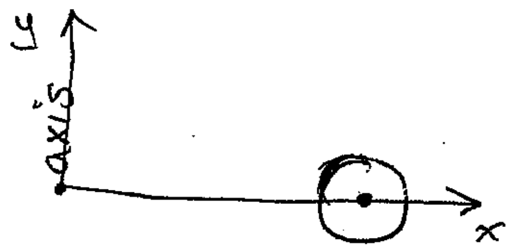
Substitute $u = e^{-x}$

$du = -e^{-x}\, dx$

$SA = 2\pi \int_1^0 -\sqrt{1+u^2}\, du = 2\pi \int_0^1 \sqrt{1+u^2}\, du$

$= 2\pi \left[\frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2}\ln\left(\sqrt{1+u^2} + u\right) \Big|_0^1 \right]$

$= 2\pi \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2} + 1) \right]$

10. 

Torus:
The circle of radius r centered at (R, 0) is rotated about the y-axis. Find the surface area.

Parameterize the circle as $x = R + r\cos t$
$y = r\sin t \qquad 0 \le t \le 2\pi$

The distance to the axis is $x = R + r\cos t$ and
$dL = \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = r\,dt$

$$SA = 2\pi \int_0^{2\pi} (R + r\cos t)\, r\, dt$$

$$= 2\pi \left[Rrt + r^2\sin t \Big|_0^{2\pi} \right]$$

$$= 4\pi^2 Rr$$

Using the thm of Pappus; by symmetry the centroid is (R, 0).

The circumference × distance traveled by the centroid is $(2\pi r)(2\pi R) = 4\pi^2 Rr$.