Math 152 10.6
Functions as Power Series
Additional Examples

Ex. We know \( f(x) = \frac{1}{1-x} \) has domain 
\((\infty, 1) \cup (1, \infty)\),
but on the interval \((-1, 1)\) we have

\[ f(x) = \sum_{n=0}^{\infty} x^n, \text{ (from the geometric series) } \]

What other functions can we represent as power series using this information?

These examples are power series about \( a = 0 \), found by substituting \( cx^b \) for \( x \) where \( b \) is a positive integer.

a) \[ \frac{1}{1+x} = \frac{1}{1-(\frac{-x}{1})} = \sum_{n=0}^{\infty} \left( \frac{-x}{1} \right)^n = \sum_{n=0}^{\infty} (-1)^n x^n \]
for \( |-x| < 1 \) or \( |x| < 1 \), interval \((-1, 1)\)

b) \[ \frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} (\frac{x}{2})^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} \]
\[ = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \text{ for } \left| \frac{x}{2} \right| < 1 \text{ or } |x| < 2 \]
The radius is \( R = 2 \), interval \((-2, 2)\)
c) \[ \frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} \text{ on } (-1, 1). \]

d) \[ \frac{1}{1+x} = \frac{1}{4(1+x/4)} = \frac{1}{4(1-(x/4))} = \frac{1}{4} \sum_{n=0}^{\infty} (-x/4)^n \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^n \text{ for } |x| < 4 \quad R = 4 \]

Now plug in \( x^2 \):

c) \[ \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n} \text{ for } |x^2| < 4 \]
\[ \text{or } |x| < 2 \quad R = 2 \]

We can also multiply a power series about 0 by a whole positive power of \( x \) and still have a power series about 0.

d) \[ \frac{x^2}{1-x} = x^2 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2} = \sum_{n=2}^{\infty} x^n \]
\[ \text{re-indexed} \]
2) Power Series about a where \( a \neq 0 \).

a) \( \frac{1}{1-x} \) about \( a = 2 \)

We need powers of \((x-2)\), that is

\[ \sum_{n=0}^{\infty} c_n (x-2)^n \]

We want to substitute into \( \sum_{n=0}^{\infty} u^n \).

\[
1-x = \frac{1}{1} - \frac{(x-2)}{1} = -1 - (x-2) = -(1 - \frac{x-2}{1})
\]

Now let \( u = -(x-2) \) and multiply the whole series by \(-1\).

\[
\frac{1}{1-x} = - \sum_{n=0}^{\infty} \left( \frac{x-2}{-1} \right)^n = \sum_{n=0}^{\infty} (-1)^n (x-2)^n
\]

\[
= \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n
\]

Note: \((-1)^{n+1} = (-1)^{n-1}\) so either can be used,
3] These examples involve taking derivatives or antiderivatives, \( a = 0 \)

\( a) \frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) \) check it!

Don't forget the chain rule.

\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx} \left( \frac{1}{1+x+x^2+\ldots} \right)
\]

\[
= \sum_{n=1}^{\infty} nx^{n-1}
\]

Note: \( n \) starts at 1 since \( \frac{d}{dx} (1) = 0 \)

\( b) \frac{x^2}{(1-x)^2} = x^2 \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n+1} \)

By re-indexing \( \sum_{n=0}^{\infty} (n+1)x^n \) on \((-1, 1)\).

\( c) \ln(1+x^2) = f(x) \), Order is important. First find the power series about \( a = 0 \) for \( \frac{1}{1+u} \). Then find \( \ln(1+u) \).

Lastly, substitute \( u = x^2 \) into the series for \( \ln(1+u) \).
c) continued
\[ \frac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n \quad \text{on } (-1, 1). \]

\[ \ln(1+u) = \int \frac{1}{1+u} \, du = \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{n+1} + C \]
(by thm.)

\[ \ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n+1} + C = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} + C \]
reindexed
\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} + C \quad \text{on } (-1, 1) \]

Find \( C \): The series is \( 0 + C \) at \( x = 0 \), and \( \ln(1+0) = 0 \) so \( C = 0 \).

Compare the order in this example to the following example.

d) \[ \arctan x = \int \frac{1}{1+x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n} \, dx \]

\[ = \sum_{n=0}^{\infty} x^{2n} \frac{(-1)^n}{n} \, dx \]
by thm.
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C \quad \text{but } C = 0 \quad \text{since} \]
\[ \arctan(0) = 0. \]
e) \( \ln(4+x) \) In this case, \( C \neq 0 \).

\[
\ln(4+x) = \int \frac{1}{4+x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^{n+1}} 
\]

by thm

\[
\frac{\sum (-1)^n x^{n+1}}{(n+1)4^{n+1}} + C = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n4^n}
\]

reindexing

Substituting \( x = 0 \)

\[
\ln 4 = C
\]

\[
\ln(4+x) = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n4^n} \text{ for } |x| < 4
\]

\((-4, 4)\)

and possibly at \(-4\) or \(4\)

The interval of convergence is \([-4, 4]\)

Check it!