7.2 More Volumes by Slices

Solutions

1. Rotated about the x-axis

\[ V = \pi \int_0^4 (x^2)^2 \, dx = \pi \int_0^4 x^3 \, dx = \left[ \frac{x^4}{4} \right]_0^4 = 64\pi \]

2. Rotated about the y-axis

To use washers, we solve for \( x_1(y) \) and \( x_2(y) \) and find

\[ \pi \int_0^1 (x_1(y))^2 - (x_2(y))^2 \, dy \]

\[ y = 2x - x^2 = -(x^2 - 2x) = -(x-1)^2 + 1 \]

\[ y - 1 = -(x-1)^2 \quad \frac{1 - y}{(x-1)^2} = \frac{\pm \sqrt{1-y} + 1}{x} \]

\[ x_1(y) = 1 + \sqrt{1-y} \quad x_2(y) = 1 - \sqrt{1-y} \]

\[ \pi \int_0^1 \left(1 + \sqrt{1-y}\right)^2 - \left(1 - \sqrt{1-y}\right)^2 \, dy \quad \text{**See Note} \]

\[ = \pi \int_0^1 4\sqrt{1-y} \, dy = \pi \left[ -\frac{8}{3} (1-y)^{3/2} \right]_0^1 \]

\[ = \frac{8}{3} \pi \]

**Note**

\[ (a+b)^2 - (a-b)^2 = 4ab \]
3. \[ y = x, \quad x = 4, \quad x = 2, \quad x = y \]
and \( y = 0 \) about \( x = 1 \).

Outer Radius is always \( 4 - 1 = 3 \)
Inner Radius is \( y - 1 \) for \( 2 \leq y \leq 4 \)
Inner Radius is \( 1 \) for \( 0 \leq y \leq 2 \)

\[ V = \pi \int_{0}^{2} 3 - y^2 \, dy + \pi \int_{2}^{4} 3 - (y - 1)^2 \, dy \]

\[ = \pi \int_{0}^{2} 8 \, dy + \pi \int_{2}^{4} 2 - y^2 + 2y \, dy \]

\[ = 16\pi + \left[ \pi \left( 8y - \frac{1}{3} y^3 + y^2 \right) \right]_{2}^{4} \]

\[ = 16\pi + \pi \left[ 16 - \frac{1}{3} (64 - 8) + (16 - 4) \right] \]

\[ = \frac{25}{3} \pi \text{ or } \frac{76}{3} \pi \]
The side of the square at \( x \) is \( 2y \), so \( A(x) = 4y^2 = 4(9-x^2) \).

\[
\int_{-3}^{3} A(x) \, dx = 2 \int_{0}^{3} A(x) \, dx
\]

by symmetry

\[
= 8 \int_{0}^{3} (9-x^2) \, dx = 8 \left[ 9x - \frac{1}{3}x^3 \right]_{0}^{3}
\]

\[
= 8 \left[ 27 - 9 \right] = 144 \text{ cu. units}
\]
The shaded region rotated about the y-axis forms the cap of the sphere.

\[ V = \pi \int_{r-h}^{r} r^2 - y^2 \, dy \]

\[ = \pi \int_{r-h}^{r} r^2 \, dy - \pi \int_{r-h}^{r} y^2 \, dy \]

\[ = \pi r^2 (h) - \frac{1}{3} \pi \left[ r^3 - (r-h)^3 \right] \]

\[ = \pi r^2 h - \frac{1}{3} \pi \left[ r^3 - (r^3 - 3r^2 h + 3rh^2 - h^3) \right] \]

\[ = \pi r^2 h - \frac{1}{3} \pi \left[ 3rh^2 - \frac{1}{3} h^3 \right] \]

\[ = \pi h^2 (r - \frac{1}{3} h) \]