Math 152  Section 10.2  Series

For a given sequence \( \{a_n\}_{n=1}^{\infty} \), \( \sum_{n=1}^{\infty} a_n \) or \( \sum_{n=1}^{\infty} a_n \) is the sequence, \( \{s_n\} \) where

\[
s_n = \sum_{k=1}^{n} a_k \text{ is the sum of the first } n \text{ terms of the given sequence, } \{a_n\}.
\]

\( s_n \) is called the \( n \)th partial sum.

Example: \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is the notation for the sequence of partial sums \( \{s_n\} \)

\[
s_1 = 1, \quad s_2 = 1 + \frac{1}{4}, \quad s_3 = 1 + \frac{1}{4} + \frac{1}{9}, \quad s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}, \ldots
\]

If \( \lim_{n \to \infty} s_n \) exists and is equal to \( S \), the series converges to \( S \) and \( \sum_{n=1}^{\infty} a_n = S \).

If \( \lim_{n \to \infty} s_n \) does not exist, the series diverges.

Example 1: Show that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.
Example 2:
The Geometric Series, \( \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots \) converges if and only if \( |r| < 1 \).

If \( |r| < 1 \) then \( \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \).

Example 3: Find the sum of the series \( \sum_{n=0}^{\infty} \left( \frac{2}{5} \right)^n = \sum_{n=1}^{\infty} \left( \frac{2}{5} \right)^{n-1} \).

Example 4: Find the sum of the series \( \sum_{n=0}^{\infty} \frac{5 \cdot 3^n}{4^n} \).
Some series are collapsing, also called telescoping.

Example: \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) Consider the general term, \( \frac{1}{n(n+1)} \).

\[
\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}
\]

We can write;

\[
s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1}
\]

which collapses into \( 1 - \frac{1}{n+1} \) which converges to 1. This means the series converges and its sum is 1.

Example 5: Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n} \).

Theorem: If \( \sum_{n=1}^{\infty} a_n \) converges then \( \lim_{n \to \infty} a_n = 0 \).

The Divergence Test: If \( \lim_{n \to \infty} a_n \neq 0 \) then the series \( \sum_{n=1}^{\infty} a_n \) diverges.
Examples: \( \sum \frac{n}{n + 2} \) diverges since the general term converges to 1, not 0.

\( \sum (-1)^n \) diverges since the general term does not converge.

Linear Property of Convergent Series:

If \( \sum_{n=1}^{\infty} a_n = A \) and \( \sum_{n=1}^{\infty} b_n = B \) then \( \sum_{n=1}^{\infty} (c \cdot a_n + d \cdot b_n) = cA + dB \).

Example 6: Find the sum of the series \( \sum_{n=1}^{\infty} \frac{5 \cdot 2^n + 2 \cdot 3^n}{4^n} \).