Math 152 notes Section 10.4  Alternating Series, Absolute convergence, Ratio Test

Any series in which the sign of the general terms alternates from positive to negative is an alternating series. If \( \{b_n\} \) is a sequence of positive terms then \( \sum (-1)^n b_n \) is an alternating series.

Example:

\[
\sum (-1)^n \frac{1}{n}
\]

is alternating.

\[
\sum (-1)^n \cos n, \quad \sum \frac{\sin n}{n}
\]

are not alternating.

Alternating Series Test: If \( \{b_n\} \) is a sequence of positive terms decreasing to 0, then \( \sum (-1)^n b_n \) converges.

Example 1. Show that \( \sum_{n=1}^{\infty} (-1)^n \sin(1/n) \) converges.

Example 2. Show that \( \sum (-1)^n \left(\frac{1}{n^{1/n}} - 1\right) \) converges.
If the sequence \( \{ b_n \} \) is not decreasing, the alternating series test does not apply and the series might or might not converge as seen in the following examples.

Example 3. Show that the series

\[
\sum_{n=1}^{\infty} a_n, \quad a_n = \begin{cases} 
\frac{1}{n} & \text{if } n \text{ is odd} \\
\frac{-1}{n^2} & \text{if } n \text{ is even}
\end{cases}
\]

is diverges.

Example 4. Show that the series

\[
\sum_{n=1}^{\infty} a_n, \quad a_n = \begin{cases} 
\frac{1}{n^2} & \text{if } n \text{ is odd} \\
\frac{-1}{n^3} & \text{if } n \text{ is even}
\end{cases}
\]

converges.

Remainder estimate: If \( \{b_n\} \) is a sequence of positive terms decreasing to 0, then

\[
|R_n| \leq \sum_{k=n+1}^{\infty} (-1)^k b_k \leq b_{n+1}.
\]

Example 4. Estimate the remainder, \( R_6 \), for the series

\[
\sum_{n=2}^{\infty} \frac{1}{n \ln n}.
\]
Absolute convergence: Any series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

Example:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ is absolutely convergent}.$$  

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ is convergent but not absolutely convergent}$$

Example 5: Determine whether or not the series converges conditionally, absolutely or neither.

a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$  

b) $\sum_{n=1}^{\infty} (-1)^n \sin \left(\frac{1}{n}\right)$

Ratio Test: If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and is equal to $L$ then $\sum a_n$:

I converges absolutely if $L < 1$.

II diverges if $L > 1$.

III no conclusion if $L = 1$. 
Example 6: Determine the conclusion of the ratio test in each case or state no conclusion.

a) \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \] Note: \[ n! = n(n-1)(n-2)(n-3)\ldots 2, \quad 1! = 1, \quad 0! = 1 \]

b) \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \]

c) \[ \sum_{n=2}^{\infty} \cos \left( \frac{n \pi}{2n} \right) \]