Volumes by Cylindrical Shells

When the region bounded by \( y = f(x) \), \( y = 0 \), \( x = a \) and \( x = b \) is rotated about the \( y \)-axis, the volume of the solid formed is

\[
V = 2\pi \int_a^b x f(x) \, dx
\]

The strip at \( x \) of width \( \Delta x \) rotates around the \( y \)-axis making a hollow cylinder of radius \( x \) and height \( f(x) \) and volume \( dV \sim 2\pi x f(x) \Delta x \)

Examples. Find the volume of the solid formed when the region described is rotated about the \( y \)-axis.

1. \( R \) is bounded by \( y = x^2 \), \( y = x \) and \( x = 2 \).

\[
V = 2\pi \int_1^2 x (x^2 - x) \, dx
\]
2. \( R \) is odd, by \( y = e^{-x^2}, y = 0, x = 0, x = 1 \).
\[ V = 2\pi \int_{0}^{1} x e^{-x^2} \, dx \]

3. \( R \) is odd, by \( y = \frac{\ln x}{x^2}, x = 1, \) and \( x = 2 \).
\[ V = 2\pi \int_{1}^{2} x \frac{\ln x}{x^2} \, dx = 2\pi \int_{1}^{2} \frac{\ln x}{x} \, dx \]

4. \( R \) is odd, by \( y = \ln x, y = 0, x = 4 \).
\[ V = 2\pi \int_{1}^{4} x \ln x \, dx \]

We need integration by parts, which comes later in the course.
Set it up using washers.
\[
V = \pi \int_{0}^{\ln 4} \left( \frac{1}{4} - (e^y)^2 \right) dy \\
= \pi \int_{0}^{\ln 4} \frac{1}{16} - e^{2y} dy
\]

5. R is odd by \( y = x^2 \), \( y = x \), \( x = 2 \) but is rotated about the vertical line \( x = -2 \).

The radius of the cylinder at \( x \) is 
\[ x - (-2) = x + 2 \]

\[
V = 2\pi \int_{1}^{2} (x+2)(x^2-x) \, dx
\]

II Find the volume by washers or cylindrical shells:

6. R is odd by \( y^2 - x^2 = 1 \), \( x = 2 \), \( x = 0 \)
   a) and is rotated about the y-axis.
6b. The top half of \( R \) is rotated about the \( x \)-axis.

7. \( R \) is the region in the first quadrant bounded by \( x^2 + y = 4 \), \( x = 0 \), and \( y = 0 \).
   a) rotated about the \( y \)-axis
   b) rotated about the \( x \)-axis

8. \( R \) is bounded by \( y = e^x \), \( y = x \), \( x = 0 \) and \( x = 1 \).
   a) rotated about the \( x \)-axis
   b) rotated about the \( y \)-axis.