We will integrate rational functions by decomposing them into manageable forms.

To determine the decomposition, we need some algebra facts.

1. The only irreducible polynomials are linear, \( x - a \), and irreducible quadratics, \( (x - a)^2 + b^2 \).

Examples:

a) \( x^3 - 2x^2 + 9x - 18 = (x - 2)(x^2 + 9) \)

b) \( (x^2 - 3x + 2)^2 (x^2 + 2x + 2) = (x - 1)^2 (x - 2)^2 ((x + 1)^2 + 1) \)

2. Any rational function in which the numerator degree is equal to or greater than the denominator degree can be as a polynomial plus a rational function in which the numerator degree is strictly less than the denominator degree.

3. Any rational function, \( \frac{r(x)}{q(x)} \) in which the numerator degree is strictly less than the denominator degree, \( \deg(r(x)) < \deg(q(x)) \), can be decomposed as a sum of rational terms in which:

a) the denominator divides \( q(x) \) and is a power of an irreducible term.

b) the numerator has lower degree than the irreducible term in the denominator.

Examples: 1. \( \frac{x}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} \) Then we solve for \( A \) and \( B \).

2. \( \frac{x^2 + 3x + 5}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2} \)

3. \( \frac{x^2 + 5x}{(x + 1)^2} = \frac{p(x)}{(x + 1)^2} + \frac{r(x)}{(x + 1)^2} \), \( \deg(r(x)) < 2 \)

\( = p(x) + \frac{A}{x + 1} + \frac{B}{(x + 1)^2} \)
Examples:

Work out each of examples 1, 2 and 3 above.

Find the decomposition and evaluate:

4. \[ \int \frac{x^2 + 3x + 5}{(x^2 + 2)^2} \, dx \]

5. \[ \int \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x + 1} \, dx \]

6. \[ \int \frac{3x^2 + x + 1}{(x - 1)(x^2 + 4)} \, dx \]