Week in Review 152
10.2 Solutions

1a) $\frac{1}{(n+3)(n+4)} = \frac{1}{n+3} - \frac{1}{n+4}$

$a_1 = \frac{1}{4} - \frac{1}{5}$

$a_2 = \frac{1}{5} - \frac{1}{6}$

$a_3 = \frac{1}{6} - \frac{1}{7}$

$\sum_{n=1}^{\infty} a_n = \frac{1}{4} - \lim_{n\to\infty} \frac{1}{n+4} = \frac{1}{4}$

1b) $\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$

$\sum_{n=1}^{\infty} a_n = \frac{1}{2}\left[1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots\right.$

$= \frac{1}{2}\left[1 + \frac{1}{2}\right] = 3/4$

1c) $5\left[1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots\right]$

$= 5\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 5\,\frac{1}{1-2/3} = 15$

1d) $\sum_{n=0}^{\infty} 3\cdot\frac{2^n}{5^{n+1}} = \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{3}{5}\left(\frac{1}{1-2/5}\right) = 1$

$\frac{3}{5}\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n-1}$

2. $S_n = 5 - \frac{n}{3^n}$ $\quad \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}\left(5 - \frac{n}{3^n}\right) = 5$

$$a_n = S_n - S_{n-1}$$

$$= 5 - \frac{n}{3^n} - \left(5 - \frac{n-1}{3^{n-1}}\right) = \frac{n-1}{3^{n-1}} - \frac{n}{3^n}$$

$$= \frac{1}{3^{n-1}}\left(n-1-\frac{n}{3}\right) = \frac{1}{3^{n-1}}\left(\frac{2}{3}n - 1\right)$$

3, a) $\sum_{n=1}^{\infty} \frac{n+3}{n^2} > \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series and this diverges.

b) $a_n = 2^{1/n}$ $\quad \lim_{n\to\infty} 2^{\frac{1}{n}} = 1$

Since $a_n \not\to 0$, $\sum a_n$ diverges.

c) $a_n = n \sin\frac{1}{n}$ $\quad \lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$

since $\lim_{h\to 0} \frac{\sin h}{h} = 1$

Since $a_n \not\to 0$,
$\sum a_n$ diverges.

152 Week in Review 10.3, 10.4 Solutions

4 a) $n^3 - n < n^3$ and $n^2 + 3n + 1 > n^2$ so

$$\frac{n^2+3n+1}{n^3-n} > \frac{n^2}{n^3} = \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^3-n}$ diverges.

b) Since $\frac{n^2-4n-5}{n^3+2} < \frac{1}{n}$ but behaves like $\frac{1}{n}$ as $n \to \infty$, we use the limit comparison test.

$$\frac{\frac{n^2-4n-5}{n^3+2}}{\frac{1}{n}} = \frac{n^3-4n^2-5n}{n^3+2} \xrightarrow[n\to\infty]{} 1$$

Since the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ is nonzero, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{n^2-4n-5}{n^3+2}$ also diverges.

c) $\frac{1}{n^{3/2}+n^{1/2}} < \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, (p-series, $p>1$), so $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n^{1/2}}$ converges.

d) By the limit comparison test and

$$\lim_{n\to\infty} \frac{\frac{1}{n^{5/4}-n}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{n^{5/4}}{n^{5/4}-n} = \lim_{n\to\infty} \frac{1}{1-n^{-1/4}} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges; $\sum_{n=2}^{\infty} \frac{1}{n^{5/4}-n}$ converges.

e) Since $\frac{1}{n} \to 0$ $(n \to \infty)$, and $\lim_{x \to 0} \frac{\sin x}{x} = 1$,

$\lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$. So $\frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}} \sim \frac{\frac{1}{n}}{\sqrt{n}} = \frac{1}{n^{3/2}}$

Using the limit comparison test with

$b_n = \frac{1}{n^{3/2}}$ we have $\lim_{n \to \infty} \frac{\frac{\sin \frac{1}{n}}{\sqrt{n}}}{\frac{1}{n^{3/2}}}$

$= \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$ and $\sum \frac{1}{n^{3/2}}$ converges

so oth series converge.

5. Using the integral test: $\frac{\ln x}{x^p}$ is decr. for $p>0$

$I = \int_2^{\infty} \frac{\ln x}{x^p}\, dx$ If $p \neq 1$

$u = \ln x$ $\quad dv = \frac{1}{x^p}\, dx = x^{-p}\, dx$

$du = \frac{1}{x} dx$ $\quad V = \frac{1}{1-p} x^{1-p}$ $\quad v\,du = x^{-1} x^{1-p}\, dx = x^{-p}\, dx$

$$I = \frac{1}{1-p} (\ln x) x^{1-p} \Big|_2^{\infty} - \frac{1}{1-p} \int_2^{\infty} x^{-p}\, dx$$

$$= \frac{1}{1-p} (\ln x) x^{1-p} \Big|_2^{\infty} - \frac{1}{(1-p)^2} x^{1-p} \Big|_2^{\infty}$$

diverges if $p<1$, converges if $p>1$.

If $p=1$, $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by comparison to $\sum \frac{1}{n}$.

5b) $\frac{1}{n^p \ln n} < \frac{1}{n^p}$ so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converges for $p > 1$.

If $p=1$ $\int_2^{\infty} \frac{1}{x \ln x} dx = \ln(\ln x)\Big|_2^{\infty}$ diverges.

so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges for $p=1$

If $p<1$, $\frac{1}{n^p \ln n} > \frac{1}{n \ln n}$ so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$

diverges.

Note: $f(x) = \frac{1}{x^p \ln x}$ is decreasing. so the test is valid.

6.

a) $\sum_{n=1}^{\infty} \frac{n}{e^n} = \sum_{n=1}^{\infty} n e^{-n}$ $f(x) = xe^{-x}$ is decreasing on $(0, \infty)$ and

$$\int_a^{\infty} xe^{-x}\, dx = -xe^{-x}\Big|_a^{\infty} + \int_a^{\infty} e^{-x}\, dx$$

$u = x$ $dv = e^{-x}$

$du = dx$ $v = -e^{-x} dx$

$$= ae^{-a} + e^{-a} \overset{*}{=} (a+1)e^{-a}$$

with $a=1$: $= e^{-1} + e^{-1}$ converges.

so by the integral test, $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges.

$$\int_{n+1}^{\infty} xe^{-x} dx \le R_n \le \int_n^{\infty} xe^{-x} dx \overset{(\text{from } *)}{=} ne^{-n} + e^{-n} = (n+1)e^{-n}$$

(from $*$) $\parallel$

$$(n+2)e^{-(n+1)} \le R_n \le (n+1)e^{-n}$$

b) We know from 2a, the series converges.

$$\int_{n+1}^{\infty} \frac{\ln x}{x^{3/2}}\,dx \le R_n \le \int_{n}^{\infty} \frac{\ln x}{x^{3/2}}\,dx$$

Plug in $p = 3/2$ to work done in 2a.

$$\int_a^{\infty} \frac{\ln x}{x^{3/2}}\,dx = -2x^{-1/2}\ln x - 4x^{-1/2}\Big]\Big|_a^{\infty} \quad \text{for } a>1$$

$$= 2a^{-1/2}\ln a - 4a^{-1/2}$$

$$= \frac{2\ln a + 4}{\sqrt{a}}$$

$$\frac{2\ln(n+1)+4}{\sqrt{n+1}} \le R_n \le \frac{2\ln n + 4}{\sqrt{n}}$$

c) Converges by comparison to $\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$.

d) Similar but larger than $\frac{\ln n}{n^{3/2}}$, use limit comparison test.

$$\frac{\;\frac{\sqrt{n}\ln n}{n^2-n}\;}{\frac{\ln n}{n^{3/2}}} = \frac{n^2}{n^2-n} = \frac{1}{1-1/n} \xrightarrow[n\to\infty]{} 1$$

So the series converges, since $\sum \frac{\ln n}{n^{3/2}}$ converges.

7. $R_n \le \int_n^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{-1}{\ln x}\Big|_n^\infty = \frac{1}{\ln n}$

$u = \ln x$

$du = \frac{1}{x}dx$

$\int \frac{1}{u^2}du = -\frac{1}{u}$

$\frac{1}{\ln n} \le 10^{-2}$ if $\ln n \ge 10^2$, $n \ge e^{100}$.

8. a) $p>1$, $\left|\frac{\sin n}{n^p}\right| \le \frac{1}{n^p}$ so the series converges absolutely.

If $p=1$, I don't know. It's not alternating, not all same sign either.

b) $\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n=4m+1 \\ -1 & \text{if } n=4m+3 \end{cases}$

$$\sum_{n=1}^\infty \frac{\sin\left(n\frac{\pi}{2}\right)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$$

$$= \sum_{n=1}^\infty (-1)^n \frac{1}{2n-1} \quad \text{converges}$$

by the alternating series **test** and $\frac{1}{2n-1}$ is decreasing to **0**.

It does not converge absolutely since $\sum \frac{1}{n}$ diverges.

9a) $\frac{1}{\ln n}$ is decreasing to 0 as $n \to \infty$,

so by the alternating series test, (AST) the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ converges and

$$|R_n| \le \frac{1}{\ln(n+1)} .$$

b) $\sin x$ is continuous, nonnegative and increasing on $[0,1]$ and $\sin 0 = 0$, so $\sin\left(\frac{1}{n}\right)$ is decreasing to 0 as $n \to \infty$.

Series converges by AST and

$$|R_n| \le \sin\frac{1}{n+1} \le \frac{1}{n+1}$$

(for convenience)

c) $\ln(1+x)$ is positive, continuous and increasing on $[0,1]$, ; $\ln 1 = 0$
so
$\ln\left(1+\frac{1}{n}\right)$ is decreasing to 0 as $n \to \infty$
The series converges by AST and

$$|R_n| \le \ln\left(1+\frac{1}{n+1}\right) \le \frac{1}{n+1}$$

since $\ln(1+x) \le x$ for $x > 0$.

10. Test for convergence, Ratio test.

a) $\sum_{n=2}^{\infty} \frac{n!}{3\cdot 5\cdot 7\cdots(2n+1)} = \sum_{n=2}^{\infty} a_n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3\cdot 5\cdots(2n+1)} \cdot \frac{3\cdot 5\cdots(2n-1)}{n!}$$

$$= \frac{n+1}{2n+1} \longrightarrow \frac{1}{2} < 1 \quad \text{Converges.}$$

b) $a_n = \frac{r^n}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{(n+1)!} \cdot \frac{n!}{|r|^n}$

$$= \frac{|r|}{n+1} \longrightarrow 0 < 1$$

so the series converges for any r.

c) Recall $\left(1-\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-1}$ so the

series should converge since $\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(1-\frac{1}{n+1}\right)^{(n+1)^2}}{\left(1-\frac{1}{n}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^{(n+1)^2} \frac{n^{n^2}}{(n-1)^{n^2}}$$

$$= \left(\frac{n}{n+1}\right)^{2n+1}\left[\frac{(n^2)^{n^2}}{(n^2-1)^{n^2}}\right] = \left(1-\frac{1}{n+1}\right)^{2n+1}\left(1+\frac{1}{n^2-1}\right)^{n^2}$$

$$\to e^{-2}e = e^{-1} < 1$$

Alternatively $\left(1-\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e}$ so eventually all a_n's are less than $\frac{2}{e}$. Since $\frac{2}{e} < 1$, $\sum_{n=\phi}^{\infty}\left(\frac{2}{e}\right)^n$ converges so $\sum_{n=\phi}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges by comparison.