Math 152 Week in Review 10.4, 10.5 solutions.

1 a.) $p>1$, $\left|\frac{\sin n}{n^p}\right| \leq \frac{1}{n^p}$ so the series converges absolutely.

If $p=1$, I don't know. It's not alternating, not all same sign either.

1 b) $\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n=4m+1 \\ -1 & \text{if } n=4m+3 \end{cases}$

$$\sum_{n=1}^{\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n-1} \quad \text{converges}$$

by the alternating series test and $\frac{1}{2n-1}$ is decreasing to 0.

It does not converge absolutely since $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{2n-1}$ diverges.

2a) $\frac{1}{\ln n}$ is decreasing to 0 as $n \to \infty$,

so by the alternating series test, (AST) the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ converges and

$$|R_n| \le \frac{1}{\ln(n+1)} .$$

2b) $\sin x$ is continuous, nonnegative and increasing on $[0, 1]$ and $\sin 0 = 0$, so $\sin\left(\frac{1}{n}\right)$ is decreasing to 0 as $n \to \infty$.

Series Converges by AST and

$$|R_n| \le \sin \frac{1}{n+1} \le \frac{1}{n+1}$$

(for convenience)

2c) $\ln(1+x)$ is positive, continuous and increasing on $[0, 1]$, $\ln 1 = 0$

so

$\ln\left(1+\frac{1}{n}\right)$ is decreasing to 0 as $n \to \infty$

The series converges by AST and

$$|R_n| \le \ln\left(1+\frac{1}{n+1}\right) \le \frac{1}{n+1}$$

since $\ln(1+x) \le x$ for $x > 0$.

3. Test for convergence, Ratio test.

a) $\sum_{n=2}^{\infty} \frac{n!}{3\cdot5\cdot7\cdots(2n+1)} = \sum_{n=2}^{\infty} a_n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3\cdot5\cdots(2n+1)} \cdot \frac{3\cdot5\cdots(2n-1)}{n!}$$

$$= \frac{n+1}{2n+1} \longrightarrow \frac{1}{2} < 1 \quad \text{Converges.}$$

b) $a_n = \frac{r^n}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{(n+1)!} \cdot \frac{n!}{|r|^n}$

$$= \frac{|r|}{n+1} \longrightarrow 0 < 1$$

so the series converges for any r.

c) Recall $\left(1 - \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-1}$ so the

series should converge since
$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(1 - \frac{1}{n+1}\right)^{(n+1)^2}}{\left(1 - \frac{1}{n}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^{(n+1)^2} \frac{n^{n^2}}{(n-1)^{n^2}}$$

$$= \left(\frac{n}{n+1}\right)^{2n+1} \left[\frac{(n^2)^{n^2}}{(n^2-1)^{n^2}}\right] = \left(1 - \frac{1}{n+1}\right)^{2n+1} \left(1 + \frac{1}{n^2-1}\right)^{n^2}$$

$$\to e^{-2} e = e^{-1} < 1$$

Alternatively $\left(1-\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e}$ so eventually all a_n's are less than $\frac{2}{e}$. Since $\frac{2}{e} < 1$, $\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n$ converges so $\sum_{n=0}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges by comparison.

152 Week in Review

10.5

1 a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x-1|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{2^n|x-1|^n}$

$$= 2|x-1| \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 2|x-1|$$

$2|x-1| < 1 \Leftrightarrow |x-1| < \frac{1}{2} = R.$

If $x-1 = -\frac{1}{2}$, the series is $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$
$x = \frac{1}{2}$
which converges by A.S.T.

If $x-1 = \frac{1}{2}$, the series is $\sum \frac{1}{\sqrt{n}}$
$x = 3/2$
which diverges. so $I = [\frac{1}{2}, \frac{3}{2})$

b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}|x-2|^{n+1}}{n+1} \cdot \frac{n}{3^n|x-2|^n} =$

$$= 3|x-2| \frac{n}{n+1} \xrightarrow[n\to\infty]{} 3|x-2|$$

$3|x-2| < 1 \Leftrightarrow |x-2| < 1/3 = R$

If $x-2 = \frac{1}{3}$, we have $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges
$x = 7/3$

If $x-2 = -\frac{1}{3}$ the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges
$x = 5/3$

so $I = (\frac{5}{3}, \frac{7}{3}]$

c) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x+3)^{2n+2}}{2^{n+1}} \cdot \frac{2^n}{(x+3)^{2n}} = \frac{(x+3)^2}{2}$

$\frac{(x+3)^2}{2} < 1 \iff x+3 < \sqrt{2} = R$

If $x+3 = \sqrt{2}$, $\sum (-1)^n$ diverges.
If $x+3 = -\sqrt{2}$ $\sum 1$ diverges.

$I = (-3-\sqrt{2}, -3+\sqrt{2})$

d) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}}$

$= \frac{x^2}{(2n+2)(2n+1)} \xrightarrow[n\to\infty]{} 0 < 1$ for all x.

so $R = \infty$ $I = (-\infty, \infty)$

e) $\left|\frac{a_{n+1}}{a_n}\right| = |2x-5| \frac{n}{n+1} \longrightarrow |2x-5|$

$|2x-5| < 1 \iff 2|x-5/2| < 1$

$|x-5/2| < 1/2 = R$

If $x - 5/2 = \frac{1}{2}$, $2x-5 = 1$ then the series is $\sum \frac{1}{n}$, diverges.
If $x - 5/2 = -1/2$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, converges

so $I = [4, 3)$.