152 WIR 10.6, 10.7 solutions

10.6

1a) $\dfrac{1}{3+x} = \dfrac{1}{3(1+\frac{x}{3})} = \dfrac{1}{3}\sum_{n=0}^{\infty}\left(\dfrac{-x}{3}\right)^n = \dfrac{1}{3}\sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{3^n}$

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{3^{n+1}} \qquad \left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x}{3}\right| < 1$ if $|x| < 3$

Diverges at -3 and 3

b) $\dfrac{1}{(x-5)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{x-5}\right) = \dfrac{d}{dx}\left(\dfrac{1}{5-x}\right)$

$= \dfrac{d}{dx}\left(\dfrac{1}{5(1-x/5)}\right) = \dfrac{1}{5}\dfrac{d}{dx}\left(\sum_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n\right)$

$= \dfrac{1}{5}\dfrac{d}{dx}\left(\sum_{n=0}^{\infty}\dfrac{x^n}{5^n}\right) = \dfrac{1}{5}\sum_{n=1}^{\infty}\dfrac{n x^{n-1}}{5^n}$ on $(-5,5)$

$= \sum_{n=1}^{\infty}\dfrac{n x^{n-1}}{5^{n+1}}$ reindex. $\sum_{n=0}^{\infty}\dfrac{(n+1)x^n}{5^{n+2}}$

$R = 5$

c) $\dfrac{x^2}{4+x^2} = x^2\left(\dfrac{1}{4+x^2}\right) = x^2 f(u) \quad u = x^2, \quad f(u) = \dfrac{1}{4+u}$

* use in 1f) $\dfrac{1}{4+u} = \dfrac{1}{4(1+\frac{u}{4})} = \dfrac{1}{4}\sum_{n=0}^{\infty}\dfrac{(-1)^n u^n}{4^n} = \sum_{n=0}^{\infty}\dfrac{(-1)^n u^n}{4^{n+1}}$

Plug in x^2:

$\dfrac{1}{4+x^2} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{4^{n+1}}$ ** (Use this in 1d)

Multiply by x^2: $\dfrac{x^2}{4+x^2} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+2}}{4^{n+1}} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1} x^{2n}}{4^n}$

$R = 2$ since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^2}{4}\right| < 1$

If $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$

You could also say $\frac{x^2}{4+x^2} = g(u) = \frac{u}{4+u}$ $u = x^2$

d) $f(x) = \arctan\left(\frac{x}{2}\right) = 2\int \frac{1}{4+x^2}\,dx$

from 1c)

$$** \quad \frac{2}{4+x^2} = 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{n+1}}$$

$$f(x) = 2\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}\,dx$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n+1}}{(2n+1)\,4^{n+1}} + C \quad \text{on } (-2, 2)$$

Solve for C $\arctan\frac{0}{2} = 0$, $\left.\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,4^{n+1}}\right|_{x=0} = 0$

so $C = 0$

This series converges on $(-2, 2]$.

1e) $x^2 \arctan x$, Multiply the series for $\arctan x$ by x^2.

$$\arctan x = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \quad \text{on } [-1,1]$$

$$\arctan 0 = 0 = \left.\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\right|_{x=0} \quad \text{so } C=0$$

$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1} \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n-1}$$

$R=1 \quad I=[-1,1]$

If $\ln(4+x^2)=f(x^2)$; $\ln(4+u) = \int \frac{1}{4+u}\,du$

* from 1c)

$$\frac{1}{4+u} = \sum_{n=0}^{\infty} \frac{(-1)^n u^n}{4^{n+1}} \quad \text{on } (-4,4)$$

$$\int \frac{1}{4+u}\,du = \left(\sum_{n=0}^{\infty} (-1)^n \frac{u^{n+1}}{(n+1)\,4^{n+1}}\right) + C$$

$c = \ln 4$

Now plug in x^2:

$$\ln(4+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1)4^{n+1}} + \ln 4$$

Converges on $[-2,2]$ $\Big\}$ $$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{4^n} + \ln 4$$

10.7

2. a) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

b) Let $g(x) = \sqrt{x+1}$, find the series for g, then substitute x^3, then find the antiderivative.

$g(0) = 1$ $\qquad g(0) = 1$

$g'(x) = \frac{1}{2}(x+1)^{-1/2}$ $\qquad g'(0) = \frac{1}{2}$

$g''(x) = \frac{1}{2}(-\frac{1}{2})(x+1)^{-3/2} = -\frac{1}{2^2}(x+1)^{-3/2}$

$g'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{1}{2}) \cdot 3 (x+1)^{-5/2} = \frac{1}{2^3} 3 (x+1)^{-5/2}$

$g^{4}(x) = \frac{-1}{2^4} 3 \cdot 5 (x+1)^{-7/2}$

$\vdots$

* $g^{n}(x) = \frac{(-1)^{n+1}}{2^n} 3 \cdot 5 \cdots (2n-3)(x+1)^{-\frac{(2n-1)}{2}}$ $\qquad n > 1$

$g^{n}(0) = \frac{(-1)^{n+1}}{2^n} 3 \cdot 5 \cdot 7 \cdots (2n-3)$

$$g(x) = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n$$

$$g(x^3) = 1 + \frac{1}{2}x^3 + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{3 \cdot 5 \cdot 7 \cdots (2n-3)}{2^n n!} x^{3n}$$

$$f(x) = x + \frac{1}{8}x^4 + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{3 \cdot 5 \cdot 7 \cdots (2n-3)}{(3n+1)2^n n!} x^{3n+1} + C$$

Since $f(0) = 2$, $C = 2$ as all other terms are 0 at 0.

c) $x \sin x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$

(reindexing) $= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n-1)!}$

d) $\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$

e) From 2f, $\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{2 \cdot x^{2n+3}}{2n+3}$ $R=1$

$x \ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{2\ x^{2n+4}}{2n+3}$

reindexing $= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2 \cdot x^{2n+2}}{2n+1}$ $R=1$

$= \sum_{n=0}^{\infty} (-1)^n \frac{2\ x^{2n}}{2n-1}$

2f) $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ on $\mathbb{R}$.

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

$x e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!}$ on $\mathbb{R}$

2g) See part of 2b

* From 2b), $g^n(x)$, using x in place of $x+1$:

3) a) $f(x) = \sqrt{x}$ $\qquad f(\ ) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(\ ) = 1/4$

$\vdots$

$f^n(x) = \frac{(-1)^{n+1}}{2^n} 3\cdot5\cdot\ldots\cdot(2n-3)\, x^{-(2n-1)/2}$ $\qquad n \geq 2$

$f^n(4) = \frac{(-1)^{n+1}}{2^n} 3\cdot5\cdot\ldots(2n-3)\, 4^{-\frac{(2n-1)}{2}}$; $4^{\frac{-2n-1}{2}} = \frac{1}{2^{2n-1}}$

$= \frac{(-1)^{n+1} 3\cdot5\ldots(2n-3)}{2^{3n-1}}$

$$f(x) = 2 + \frac{1}{4}(x-4) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{3\cdot5\ldots(2n-3)}{2^{3n-1}\, n!}(x-4)^n$$

3) b) Here done by $c_n = \frac{f^n(a)}{n!}$, later another way.

$f(x) = \ln(1+x)$ $\qquad f(1) = \ln 2$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(1) = \frac{-1}{2^2}$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(1) = \frac{2}{2^3}$

$f^{IV}(x) = \frac{-2\cdot3}{(1+x)^4}$ $\qquad f^{IV}(1) = \frac{-2\cdot3}{2^4}$

$\vdots$

$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ $\qquad f^n(1) = \frac{(-1)^{n+1}(n-1)!}{2^n}$ $\qquad c_n = \frac{(-1)^{n+}}{n2^n}$

$$\ln(1+x) = \qquad + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n\cdot2^n}$$

3 b)

Alternatively:

$$\frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2\left[1+\frac{x-1}{2}\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{2^n}$$

$$\ln(1+x) = \int \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{2^n}\,dx$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{(n+1)2^n} + C$$

(Plug in $x =$
$C = \ln 2$

$$= \ln 2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n\,2^{n-1}}$$

$$= \ln 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n\,2^n}$$

3c) $\ln x \quad a=1$

$$\ln x = \int \frac{1}{x}\,dx$$

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \qquad |x-1|<1$$

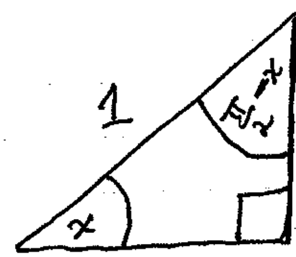
$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} + C$$

$\ln 1 = 0$ = series at $x=1$ so $C=0$

reindexing

$$\ln x = \sum_{n=}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

3d) $\sin x \quad a = \pi/2$



$\sin x = \cos(\frac{\pi}{2} - x)$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$= \cos(x - \pi/2)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

4 a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ so $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{2} = 1$

c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} = \cos 1 \qquad \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} = \cos\pi = -1$

Since both converge

$$\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n)!} + \frac{\pi^{2n}}{(2n)!}\right] = \cos 1 + \cos\pi = \cos 1 - 1$$