Week-in-Review Math 152
Solutions 6.4, 6.5

I

1) Substitute $u = 5 + x^2$

$$du = 2x\,dx$$
$$\tfrac{1}{2}\,du = x\,dx$$

$$\int x\sqrt{5+x^2}\,dx = \int \sqrt{u}\left(\tfrac{1}{2}du\right) = \tfrac{1}{2}\int \sqrt{u}\,du$$
$$= \tfrac{1}{2}\cdot\tfrac{2}{3}\cdot u^{3/2} + C = \tfrac{1}{3}(5+x^2)^{3/2} + C$$

2) $u = 4 + x^2 \qquad du = 2x\,dx \qquad \tfrac{1}{2}du = x\,dx$

$$\int \frac{5x}{4+x^2}\,dx = \int \frac{5\cdot\frac{1}{2}\,du}{u} = \frac{5}{2}\ln|u| + C$$
$$= \frac{5}{2}\ln(4+x^2) + C$$

3) $$\int \frac{12}{4+t^2}\,dt = 12\cdot\tfrac{1}{2}\arctan\frac{t}{2} + C$$
$$= 6\arctan\frac{t}{2} + C$$

4) A shift-type substitution:

$u = x - 2 \qquad du = dx$
$u + 2 = x$

$$\int (x+3)\sqrt{x-2}\,dx = \int (u+5)\sqrt{u}\,du$$
$$= \int u^{3/2} + 5u^{1/2}\,du = \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$
$$= \tfrac{2}{5}(x-2)^{5/2} + \tfrac{2}{3}(x-2)^{3/2} + C$$

5) $\int \tan\theta \, d\theta = \ln|\sec\theta| + C$
from memorized formula.

Or

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ Substitute $u = \cos\theta$
$du = -\sin\theta \, d\theta$

$$\int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{du}{u} = -\ln|u| + C$$
$$= -\ln|\cos\theta| + C$$
$$= \ln\left|\frac{1}{\cos\theta}\right| + C$$
$$= \ln|\sec\theta| + C$$

6) $u = x^2 + 6x$ $du = (2x+6)\,dx$
$= 2(x+3)\,dx$

$\frac{1}{2} du = (x+3)\,dx$

$$\int (x+3) e^{x^2+6x} dx = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + C$$
$$= \frac{1}{2} e^{x^2+6x} + C$$

7) $u = x^2$ $du = 2x\,dx$ $\frac{1}{2} du = x\,dx$

$$\int x 4^{x^2} dx = \int \frac{1}{2} 4^u \, du = \frac{1}{2} 4^u \cdot \frac{1}{\ln 4} + C$$
$$= \frac{1}{2\ln 4} 4^{x^2} + C$$

8) $u = x^2 - 4x + 3$ $\quad du = (2x-4)\,dx$

$\frac{7}{2} du = (7x - 14)\,dx$

$$\int \frac{7x-14}{(x^2-4x+3)^2}\,dx = \frac{7}{2}\int \frac{du}{u^2}$$

$$= -\frac{7}{2}\cdot\frac{1}{u} + C = -\frac{7}{2}\,\frac{1}{x^2-4x+3} + C$$

9) $\displaystyle\int \frac{4x+12}{x^2+6x+10}\,dx$ $\quad u = x^2 + 6x + 10$

$du = (2x+6)\,dx$

$2\,du = (4x+12)\,dx$

$$= \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(x^2+6x+10) + C$$

10) $u = \cos\theta$ $\quad du = -\sin\theta\,d\theta$

$$\int \sin\theta\,\cos^3\theta\,d\theta = \int -u^3\,du = -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4\theta + C$$

11) $u = \tan\theta$ $\quad du = \sec^2\theta\,d\theta$

$$\int \tan^2\theta\,\sec^2\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\tan^3\theta + C$$

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

II

12. $\displaystyle\int_0^{\sqrt{\pi/2}} x \sin x^2\,dx \qquad u = x^2 \qquad u(0)=0 \quad u(\sqrt{\pi/2}) = \pi/2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \int_0^{\pi/2} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u \Big|_0^{\pi/2}$$

$$= 0 - \left(-\frac{1}{2}\right) = \boxed{1/2}$$

13. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \Big|_0^{1/\sqrt{2}} = \frac{\pi}{4}$

14. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx \qquad u = x^2 \quad du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int_0^{1/2} \frac{1}{2}\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u \Big|_0^{1/2} = \frac{1}{2}(\pi/6) - 0$$

$$= \boxed{\pi/12}$$

15. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^3}{\sqrt{1-x^4}}\,dx \qquad u = 1-x^4 \quad u(0)=1 \quad u\left(\frac{1}{\sqrt{2}}\right) = 3/4$

$du = -4x^3\,dx$

$-\frac{1}{4}du = x^3\,dx$

$$= -\frac{1}{4}\int_1^{3/4} \frac{du}{\sqrt{u}} = -\frac{1}{4}\cdot 2\sqrt{u}\Big|_1^{3/4} = -\frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$= \boxed{\frac{1}{2} - \frac{\sqrt{3}}{4}}$$

16. $\displaystyle\int_{-2}^{2} 5x^3\,dx + \int_{-2}^{2}\sqrt{4-x^2}\,dx$

1st integral:

$$\int_{-2}^{2} 5x^3\,dx = \frac{5}{4}x^4\Big|_{-2}^{2} = 0$$

or just observe that $5x^3$ is an odd function and the interval $[-2, 2]$ is symmetric.

2nd integral is the area of the top half of the circle of radius 2 centered at $(0,0)$.
The area is $\frac{1}{2}(\pi \cdot 2^2) = 2\pi$

1st integral + 2nd integral $= \boxed{2\pi}$

17. $\displaystyle\int_{-1}^{0} -x\,dx + \int_{0}^{4} x\,dx$ Use area of Δ's or

$$= -\frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{4} =$$

$$= -\frac{1}{2}(0-1) + \frac{1}{2}(16-0) = \boxed{\frac{17}{2}}$$

18. $x^2-4x+3 = (x-3)(x-1)$

Split the integral at 1 and 3.

$$\int_0^1 x^2-4x+3\,dx + \left|\int_1^3 x^2-4x+3\,dx\right|$$

$$+\int_3^5 (x^2-4x+3)\,dx$$

$$= \tfrac{1}{3}x^3-2x^2+3x\Big|_0^1 + \left|\left(\tfrac{1}{3}x^3-2x^2+3x\right)\Big|_1^3\right|$$

$$+\left(\tfrac{1}{3}x^3-2x^2+3x\right)\Big|_3^5$$

$$= 4/3 \quad + \quad |-4/3|$$

$$+\left(\tfrac{125}{3}-50+15-\left(\overbrace{9-18+9}^{0}\right)\right)$$

$$= \frac{28}{3}$$

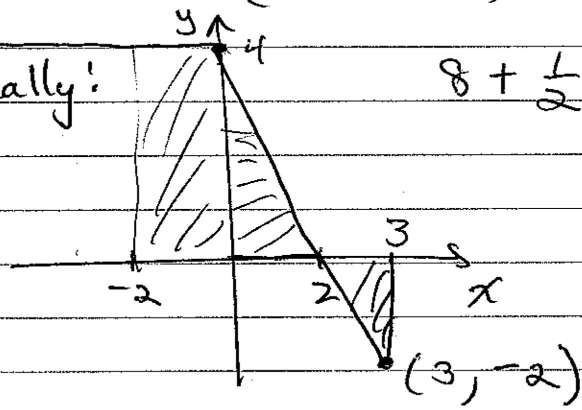
19.

$$\int_{-2}^{0} 4\,dx + \int_{0}^{3} 4-2x\,dx$$

$$= 4x\Big|_{-2}^{0} + \left(4x - x^2\Big|_{0}^{3}\right)$$

$$= 8 + (3-0) = 11$$

Geometrically:



$$8 + \tfrac{1}{2}(2\times4) - \tfrac{1}{2}(1\cdot2)$$

$$= 8 + 4 - 1 = 11$$