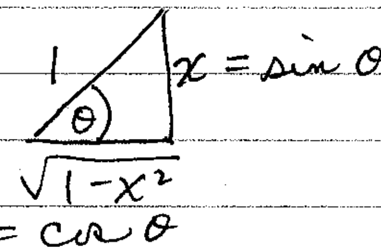
Math 152 WIR 8.3, 8.4 Solutions

8.3

1. $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

Method 1) $x = \sin\theta$

$dx = \cos\theta\, d\theta$



$$\int \frac{\sin^3\theta}{\cos\theta}(\cos\theta\, d\theta) = \int \sin^3\theta\, d\theta$$

$$= \int \sin\theta\,(1-\cos^2\theta)\,d\theta = \int (1-u^2)(-du)$$

$u = \cos\theta$

$$= \int u^2 - 1\, du = \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$$= \frac{1}{3}(1-x^2)^{3/2} - \sqrt{1-x^2} + C$$

Method 2) Substitute $u = 1 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

Then $x^3\,dx = x^2(x\,dx) = (1-u)(-\frac{1}{2}du)$

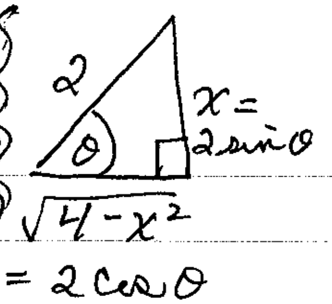
and

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du = -\frac{1}{2}\int u^{-1/2} - u^{1/2}\,du$$

$$= -u^{1/2} + \frac{1}{3}u^{3/2} + C = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C$$

2. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

$x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$



$$\int \frac{4\sin^2\theta}{(2\cos\theta)^3}(2\cos\theta\,d\theta)$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta = \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \arcsin\frac{x}{2} + C$$

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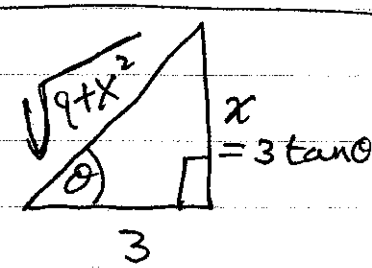
3. $\displaystyle\int \frac{x^2}{(9+x^2)^{3/2}}\,dx$

$x = 3\tan\theta$
$dx = 3\sec^2\theta\,d\theta$



$$\int \frac{9\tan^2\theta}{(3\sec\theta)^3}(3\sec^2\theta\,d\theta)$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= \int \sec\theta - \cos\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| - \frac{x}{\sqrt{9+x^2}} + C \quad \text{(see } \Delta \text{ above)}$$

$$= \ln\left|\sqrt{9+x^2} + x\right| - \frac{x}{\sqrt{9+x^2}} + \tilde{C}$$

4.a) $\displaystyle\int \frac{1}{(8+4x+x^2)^{3/2}}\,dx$

completing the square :

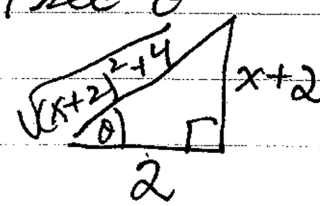
$$x^2+4x+8 = (x^2+4x+4) + 8-4$$
$$= (x+2)^2+4$$

$\displaystyle\int \frac{1}{\left[(x+2)^2+4\right]^{3/2}}\,dx$ $\qquad x+2 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$(x+2)^2+4 = 4\sec^2\theta$

$\displaystyle\int \frac{1}{(4\sec^2\theta)^{3/2}}(2\sec^2\theta)\,d\theta$

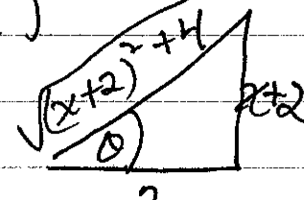


$\displaystyle = \int \frac{1}{8\sec^3\theta}\cdot 2\sec^2\theta\, d\theta = \frac{1}{4}\int \cos\theta\, d\theta$

$\displaystyle = \frac{1}{4}\sin\theta + C = \frac{1}{4}\,\frac{x+2}{\sqrt{(x+2)^2+4}} + C$

$$4b. \quad \int \frac{x}{(8+4x+x^2)^{3/2}}\,dx$$

same substitution as 4a)



$$= \int \frac{(2\tan\theta - 2)}{(2\sec\theta)^3}(2\sec^2\theta\, d\theta)$$

$$= \int \frac{\tan\theta - 1}{2\sec\theta}\, d\theta \qquad \frac{\tan\theta}{\sec\theta} = \sin\theta$$

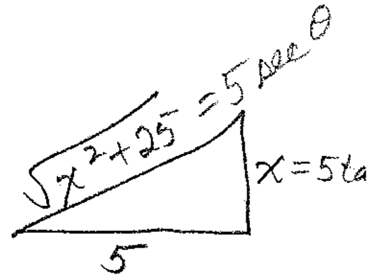
$$= \frac{1}{2}\int \sin\theta\, d\theta - \frac{1}{2}\int \cos\theta\, d\theta$$

$$= -\frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta + C$$

$$= -\frac{1}{2}\cdot\frac{2}{\sqrt{(x+2)^2+4}} - \frac{1}{2}\frac{(x+2)}{\sqrt{(x+2)^2+4}} + C$$

$$= -\frac{1}{2}\frac{x+4}{\sqrt{(x+2)^2+4}} + C$$

5. $\int \sqrt{x^2+25}\, dx \qquad x = 5\tan\theta \quad dx = 5\sec^2\theta\, d\theta$



$= \int 5\sec\theta \cdot 5\sec^2\theta\, d\theta = \int 25\sec^3\theta\, d\theta$

$\int \sec^3\theta\, d\theta$ is done by parts:

$u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$
$du = \sec\theta\tan\theta\, d\theta \quad v = \tan\theta$

$uv - \int v\, du = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$

$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$

$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$

$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

Then

$\int \sqrt{x^2+25}\, dx = 25\int \sec^3\theta\, d\theta$

$= \frac{25}{2}\sec\theta\tan\theta + \frac{25}{2}\ln|\sec\theta + \tan\theta| + C$

$= \frac{1}{2}x\sqrt{x^2+25} + \frac{25}{2}\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C$

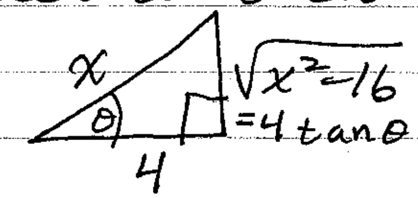
$= \frac{1}{2}x\sqrt{x^2+25} + \frac{25}{2}\ln\left|\sqrt{x^2+25} + x\right| + C_2$

6. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1\ (4\sec\theta\tan\theta\,d\theta)}{16\sec^2\theta(4\tan\theta)}$$



Make the Δ

So $\frac{4}{x} = \cos\theta$

$$= \int \frac{1}{16\sec\theta}\,d\theta$$

$$= \int \frac{1}{16}\cos\theta\,d\theta = \frac{1}{16}\sin\theta + C$$

$$= \frac{1}{16}\,\frac{\sqrt{x^2-16}}{x} + C$$

8.4 Partial Fractions

7. Factor $x^2+3x+2 = (x+2)(x+1)$

$$\int \frac{1}{(x+2)(x+1)}\,dx = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx$$

Note: $\frac{1}{(x+n)(x+n+1)} = \frac{1}{x+n} - \frac{1}{x+n+1}$

or

$$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x+2)$$

coeff. of x: $A+B=0$

constant : $A+2B=1$

$$\left.\begin{array}{r} A+2B=1 \\ A+B=0 \\ \hline 0+B=1 \end{array}\right\rangle \text{subtract}$$

and then $A=-1$

$$\int \frac{-1}{x+2} + \frac{1}{x+1}\,dx = -\ln|x+2| + \ln|x+1| + C$$

8. $\displaystyle\int \frac{1}{(x^2+4)(x+1)}\,dx$

$$\frac{1}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

$$1 = (Ax+B)(x+1) + C(x^2+4)$$

coeff. of x^2: $0 = A + C \longrightarrow A = -C$

coeff. of x: $0 = A + B \longrightarrow 0 = -C + B$

constant: $1 = B + 4C$

$$\left.\begin{array}{l} B + 4C = 1 \\ \underline{B - C = 0} \end{array}\right\} \text{subtract}$$

$0 + 5C = 1 \qquad C = \frac{1}{5} \quad B = \frac{1}{5} \quad A = -\frac{1}{5}$

$$\int \frac{1}{(x^2+4)(x+1)}\,dx = \frac{1}{5}\int \frac{-x+1}{x^2+4}\,dx + \frac{1}{5}\int \frac{1}{x+1}\,dx$$

$$= -\frac{1}{5}\int \frac{x}{x^2+4}\,dx + \frac{1}{5}\int \frac{1}{x^2+4}\,dx + \frac{1}{5}\int \frac{1}{x+1}\,dx$$

subst. $u = x^2+4$

$$= -\frac{1}{10}\ln(x^2+4) + \frac{1}{10}\arctan\frac{x}{2}$$

$$+ \frac{1}{5}\ln|x+1| + C$$

9. $$\frac{x^2+3x+16}{(x^2+9)(x-1)} = \frac{Ax+B}{x^2+9} + \frac{C}{x-1}$$

$$x^2+3x+16 = (Ax+B)(x-1) + C(x^2+9)$$

Coeff of x^2 : $1 = A + C$

of x : $3 = B - A$

of x^0 ; $16 = -B + 9C$

Adding all 3 equations gives $20 = 10C$

$\boxed{2 = C}$

substitituting $C = 2$

$A + 2 = 1$ so $\boxed{A = -1}$ $\longrightarrow$ $B + 1 = 3$

so $\boxed{B = 2}$

Plugging in :

$$\int \frac{-x+2}{(x^2+9)}\,dx + \int \frac{2}{x-1}\,dx$$

$$= \int \frac{-x}{x^2+9}\,dx + \int \frac{2}{x^2+9}\,dx + \int \frac{2}{x-1}\,dx$$

$$= -\frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\frac{x}{3} + 2\ln|x-1| + C$$

by substituting

$u = x^2+9$

10. Factor the denominator: $x^3 - 4x^2 + 4x = x(x-2)^2$

$$\frac{3x^2 - 11x + 16}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$3x^2 - 11x + 16 = A(x-2)^2 + Bx(x-2) + Cx$$
$$= A(x^2 - 4x + 4) + B(x^2 - 2x) + Cx$$

Coeff. of x^2: $3 = A + B$

of x: $-11 = -4A - 2B + C$

of x^0: $16 = 4A$ so $\boxed{A = 4}$

$3 = 4 + B$ so $\boxed{B = -1}$ and $-11 = -16 + 2 + C$
$-11 = -14 + C$
$\boxed{3 = C}$

$$\int \frac{4}{x}\,dx - \int \frac{1}{x-2}\,dx + 3\int \frac{1}{(x-2)^2}\,dx$$

$$= 4\ln|x| - \ln|x-2| - \frac{3}{(x-2)} + C$$

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11. Since the numerator does not have lower degree we must divide or

$$\frac{x^2 - 3x + 10}{x^2 - 2x + 10} = 1 - \frac{x}{x^2 - 2x + 10}$$

$$\int \left(1 - \frac{x}{x^2 - 2x + 10}\right) dx = \int dx - \int \frac{x}{(x-1)^2 + 9}\,dx$$ The 2n

term is already decomposed and

$$\frac{x}{(x-1)^2+9} = \frac{x-1}{(x-1)^2+9} + \frac{1}{(x-1)^2+9}$$

so $$\int \frac{x}{(x-1)^2+9}\,dx = \int \frac{x-1}{(x-1)^2+9}\,dx + \int \frac{1}{(x-1)^2+9}\,dx$$

$$= \frac{1}{2}\ln\left((x-1)^2+9\right) - \frac{1}{3}\arctan\frac{x-1}{3} +$$

The final result is:

$$\int 1 - \frac{x}{(x-1)^2+9}\,dx = x - \frac{1}{2}\ln\left((x-1)^2+9\right) - \frac{1}{3}\arctan\frac{x-1}{3}$$

$$+ C$$