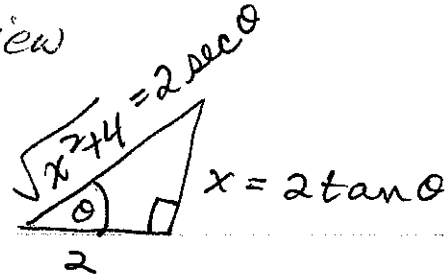
152 Exam 2 Review
Solutions

8.3

1a) $x = 2\tan\theta$



$dx = 2\sec^2\theta \, d\theta$

$$\int \frac{5x^2}{x^2+4}\,dx = \int \frac{5\cdot 4\tan^2\theta}{4\cancel{\sec^2\theta}} \cdot 2\cancel{\sec^2\theta}\, d\theta$$

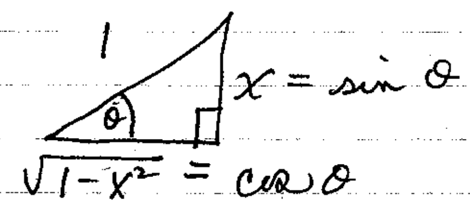
$$= 10\int \tan^2\theta \, d\theta = 10\int \sec^2\theta - 1 \, d\theta$$

$$= 10\tan\theta - 10\theta + C$$
$$= 5x - 10\arctan\frac{x}{2} + C$$

Alternatively: $\dfrac{x^2}{x^2+4} = \dfrac{x^2+4-4}{x^2+4} = 1 - \dfrac{4}{x^2+4}$

$$5\int 1 - \frac{4}{x^2+4}\,dx = 5\left[x - 4\cdot\frac{1}{2}\arctan\frac{x}{2}\right] + C$$
$$= 5x - 10\arctan\frac{x}{2} + C$$

1b) $x = \sin\theta$



$dx = \cos\theta \, d\theta$

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{\cos^3\theta}\cos\theta\,d\theta$$

$$= \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$$
$$= \frac{x}{\sqrt{1-x^2}} + C$$

8.3

1 c) $x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$\int_0^{\pi/2} 8\cos^2\theta\, d\theta = \int_0^{\pi/2} 8\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 4\theta + 4\cdot\tfrac{1}{2}\sin 2\theta \Big|_0^{\pi/2} = 4\left(\frac{\pi}{2}\right) + 0 - 0$$

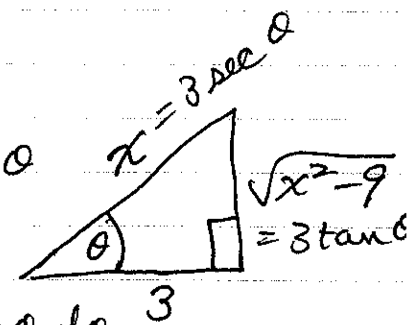
$$= 2\pi$$

Alternatively $y = \sqrt{1-x^2}$ is the top of the unit circle. If $0 \le x \le 1$, we have $\frac{1}{4}$ of the unit circle.

$$\int_0^1 8\sqrt{1-x^2}\, dx = 8\cdot\left(\frac{\pi}{4}\right) = 2\pi$$

1 d) $x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta$



$$\int \frac{\sqrt{x^2-9}}{x}\, dx = \int \frac{3\tan\theta}{\cancel{3\sec\theta}}\cdot \cancel{3\sec\theta}\tan\theta\, d\theta$$

$$= \int 3\tan^2\theta\, d\theta = \int 3(\sec^2\theta - 1)\, d\theta = 3\tan\theta - 3\theta + C$$

$$= \sqrt{x^2-9} - 3\arctan\frac{\sqrt{x^2-9}}{3} + C$$

or $\sqrt{x^2-9} - 3\,\mathrm{arcsec}\,\frac{x}{3} + C$

1e) $x^2+4x+13 = x^2+4x+4+9$
$= (x+2)^2+9$

$x+2 = 3\tan\theta \qquad \sqrt{(x+2)^2+9} = 3\sec\theta$
$dx = 3\sec^2\theta\, d\theta$

$$\int \frac{6}{\sqrt{(x+2)^2+9}}\,dx = \int \frac{6}{\cancel{3}\,\cancel{\sec\theta}} \cdot \cancel{3}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \int 6\sec\theta\, d\theta = 6\ln|\sec\theta+\tan\theta| + C$$

$$= 6\ln\left|\sqrt{x^2+4x+13} + x+2\right| + C_1$$

where $C_1 = C - 6\ln 3$

8.4

2a) Since the numerator degree is higher than the denominator degree, we divide.

$$x^3-4x^2 = x(x^2-4x) = x(x^2-4x+13-13)$$
$$= x(x^2-4x+13) - 13x$$

$$\frac{x^3-4x^2}{x^2-4x+13} = x - \frac{13x}{x^2-4x+13} = x - \frac{13x}{(x-2)^2+9}$$

$(x-2)^2+9$ is irreducible

It is in the partial fractions decomposition.

$$x - \frac{13x}{(x-2)^2+9} = x - \frac{13(x-2)}{(x-2)^2+9} - \frac{26}{(x-2)^2+9}$$

$$\int x - \frac{13(x-2)}{(x-2)^2+9}\,dx - \int \frac{26}{(x-2)^2+9}\,dx$$

$$= \frac{1}{2}x^2 - \frac{13}{2}\ln\left((x-2)^2+9\right) - \frac{26}{3}\arctan\left(\frac{x-2}{3}\right) + C$$

2 b) $\int \frac{1}{x(x+2)^2}\,dx$

$$\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} = \frac{1}{x(x+2)^2}$$

$A(x+2)^2 + Bx(x+2) + Cx = 1$

Substituting $x=0$ gives $4A = 1$, $A = 1/4$.

Comparing coefficients:

x^2: $A + B = 0$ so $B = -1/4$

x : $4A + 2B + C = 0$ so $1 - \frac{1}{2} + C = 0$

$C = -1/2$

$$\int \frac{1}{x(x+2)^2}\,dx = \int \frac{1}{4x}\,dx - \int \frac{1}{4(x+2)}\,dx - \int \frac{1}{2(x+2)^2}\,dx$$

$$= \frac{1}{4}\ln|x| - \frac{1}{4}\ln|x+2| + \frac{1}{2}\cdot\frac{1}{(x+2)} + C$$

2 c) $\displaystyle\int \frac{x^2+1}{(x^2+4)(x+1)}\,dx$

$$\frac{Ax+B}{x^2+4} + \frac{C}{x+1} = \frac{x^2+1}{(x^2+4)(x+1)}$$

$$(Ax+B)(x+1) + C(x^2+4) = x^2+1$$

Coeff of x^2 : $A + C = 1$ $\qquad -B + C = 1$

x : $A + B = 0 \qquad A = -B$

x^0 : $B + 4C = 1 \qquad B + 4C = 1$

add $\qquad 0 + 5C = 2$

$C = 2/5$

$C = 2/5$ and $B = C - 1$ so $B = -3/5$

$A = -B \quad A = 3/5$

$$\frac{3}{5}\int \frac{x}{x^2+4}\,dx - \frac{3}{5}\int \frac{1}{x^2+4}\,dx + \frac{2}{5}\int \frac{1}{x+1}\,dx$$

$$= \frac{3}{5}\cdot\frac{1}{2}\ln(x^2+4) - \frac{3}{10}\arctan\frac{x}{2} + \frac{2}{5}\ln|x+1| + C$$

8.9

3.a) $\frac{1}{x+\sqrt{x}} \geq \frac{1}{2x}$ since $x+\sqrt{x} \leq 2x$ for $x \geq 1$.

$$\int_1^\infty \frac{1}{x+\sqrt{x}}\,dx \geq \lim_{T\to\infty} \int_1^T \frac{1}{2x}\,dx = \lim_{T\to\infty} \frac{1}{2}\ln T = \infty$$

$\boxed{\text{Diverges}}$

b) $0 \leq \frac{1}{x+\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ since $x+\sqrt{x} \geq \sqrt{x}$ for $x \geq 0$

$$\int_0^1 \frac{1}{x+\sqrt{x}}\,dx \leq \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0} (2-2\sqrt{t}) = 2$$

Converges

c) $\int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx$ $\quad$ $x^2+\sqrt{x} > x^2$

$$\frac{1}{x^2+\sqrt{x}} < \frac{1}{x^2}$$

$$\int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx < \int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty} -\frac{1}{x}\Big|_1^T$$

$$= \lim_{T\to\infty} \left(\frac{-1}{T}+1\right) = 1$$

Converges

3d) $$\int_1^T \frac{1}{x(x+1)}\,dx = \int_1^T \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \ln x - \ln(x+1) \Big|_1^T = \ln(T) - \ln(T+1) + \ln 2$$

$$= \ln\left(\frac{T}{T+1}\right) + \ln 2$$

$$\lim_{T\to\infty}\left(\ln\frac{T}{T+1} + \ln 2\right) = \ln 1 + \ln 2 = \boxed{\ln 2}$$

3e) $$\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$$

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$$= -xe^{-x} - e^{-x} + C$$

$$\lim_{T\to\infty}\int_1^T xe^{-x}\,dx = \lim_{T\to\infty}\left(-xe^{-x} - e^{-x}\Big|_1^T\right)$$

$$= \lim_{T\to\infty}\left(-Te^{-T} + e^{-T} - \left(-e^{-1} - e^{-1}\right)\right)$$

$$= 0 + 0 - (-2e^{-1}) = \boxed{2e^{-1}}$$

3f) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

$u = \ln x \qquad dv = \frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{x}\,dx \qquad v = 2\sqrt{x}$

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0}\left[2\sqrt{x}\ln x - 4\sqrt{x}\Big|_t^1\right]$$

$$= \lim_{t\to 0}\left[0 - 4 - (2\sqrt{t}\ln t - 4\sqrt{t})\right] = \boxed{-4}$$

9.3

4 a) $x = t^2 \cos t \qquad y = t^2 \sin t \qquad 0 \le t \le 2\pi$

$x'(t) = 2t\cos t - t^2 \sin t \qquad y'(t) = 2t \sin t + t^2 \cos t$

$$(x'(t))^2 + (y'(t))^2 = 4t^2\cos^2 t - 2t^3 \cos t \sin t + t^4 \sin^2 t$$

$$+ \quad 4t^2 \sin^2 t + 2t^3 \cos t \sin t + t^4 \cos^2 t$$

$$= 4t^2 + t^4$$

$$= t^2(4 + t^2)$$

$$dL = \sqrt{t^2(4+t^2)} = t\sqrt{4+t^2}$$

$$L = \int_0^{2\pi} t\sqrt{4+t^2}\, dt = \frac{1}{2}\cdot\frac{2}{3}(4+t^2)^{3/2}\Big|_0^{2\pi}$$

substitute $u = 4 + t^2$

$du = 2t\,dt$

$$= \frac{1}{3}(4+4\pi^2)^{3/2} - \frac{1}{3}(4^{3/2})$$

$$= \frac{1}{3}(4+4\pi^2)^{3/2} - 8/3$$

or $\frac{8}{3}\left((1+\pi^2)^{3/2} - 1\right)$

4b) $y'(x) = \dfrac{e^x - e^{-x}}{2}$

$$1+(y'(x))^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= 1 + \frac{e^{2x} + e^{-2x}}{4} - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{e^{2x} + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$dL = \frac{e^x + e^{-x}}{2} \qquad L = \int_0^1 \frac{e^x + e^{-x}}{2}\,dx$$

$$= \frac{e^x - e^{-x}}{2}\Big|_0^1 = \frac{e - e^{-1}}{2} - 0 = \left(\frac{e - \frac{1}{e}}{2}\right)$$

Alternatively:

$$y = \cosh x \qquad y' = \sinh x \qquad 1 + (y')^2 = \cosh^2 x$$

$$dL = \cosh x \qquad L = \int_0^1 \cosh x \, dx$$

$$= \sinh x \Big|_0^1 = \sinh 1$$

4 c) $y'(x) = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$

$$1 + (y'(x))^2 = 1 + \left(\frac{1}{4}x - \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{x}\right)$$

$$= \frac{1}{4}x + \frac{1}{2} + \frac{1}{4x}$$

$$= \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)^2$$

$$dL = \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$$

$$L = \int_0^1 \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\,dx$$

$$= \cancel{\tfrac{1}{2}}\cdot\frac{\cancel{2}}{3}x^{3/2} + x^{1/2}\Big|_0^1 = \frac{1}{3} + 1 = \boxed{4/3}$$

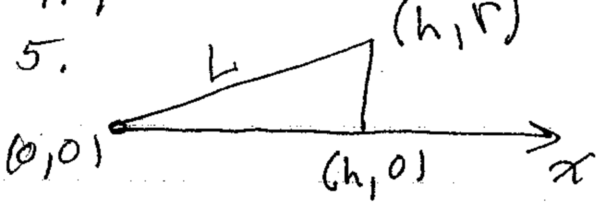
4d) Notice that
$$x^2 = \cos^2 t + 2\cos t \sin t + \sin^2 t$$
$$y^2 = \cos^2 t - 2\cos t \sin t + \sin^2 t$$
$$x^2 + y^2 = 2\cos^2 t + 2\sin^2 t$$
$$= 2$$

We want the length of $\frac{1}{2}$ of the circle of radius $\sqrt{2}$

which is $\frac{2\pi\sqrt{2}}{2} = \pi\sqrt{2}$

9.4

5.



If L is rotated about the x-axis we get the cone describled.

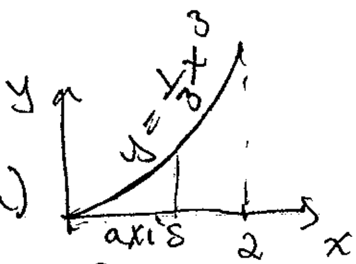
$$SA = 2\pi \int_0^h y \, dL \qquad y = \frac{r}{h} x$$

$$= 2\pi \int_0^h \frac{r}{h} x \sqrt{1 + \left(\frac{r}{h}\right)^2} \, dx$$

$$= \cancel{2}\pi \frac{r}{h} \sqrt{1 + \left(\frac{r}{h}\right)^2} \cdot \frac{1}{\cancel{2}} x^2 \Big|_0^h$$

$$= \pi \frac{r}{\cancel{h}} \sqrt{1 + \left(\frac{r}{h}\right)^2} \, h^{\cancel{2}} = \pi r \sqrt{h^2 + r^2}$$

6. a)



$$dL = \sqrt{1 + x^4} \, dx$$

$$2\pi \int_0^2 y \, dL = 2\pi \int_0^2 \frac{1}{3} x^3 \sqrt{1 + x^4} \, dx$$

$u = 1 + x^4 \quad u(0) = 1 \;\; u(2) = 1$

$du = 4x^3 dx$

$$= \frac{2\pi}{3} \int_1^{17} \frac{1}{4} \sqrt{u} \, du$$

$$= \frac{2\pi}{3} \cdot \frac{2}{3} u^{3/2} \Big|_1^{17} = \frac{4\pi}{9} \left[17^{3/2} - 1 \right]$$

6 b) $y' = \frac{1}{x}$ $dL = \sqrt{1 + \frac{1}{x^2}}\, dx$ $r = x$

$S = 2\pi \int_1^e x\sqrt{1 + \frac{1}{x^2}}\, dx$ $x\sqrt{1+\frac{1}{x^2}} = x\frac{\sqrt{x^2+1}}{x}$

$= 2\pi \int_1^e \sqrt{x^2+1}\, dx$ $x = \tan\theta$ $\sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$\int \sqrt{1+x^2}\, dx = \int \sec^3\theta\, d\theta$

$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|\sqrt{x^2+1} + x| + C$

$$S = 2\pi\left[\frac{1}{2}e\sqrt{e^2+1} + \frac{1}{2}\ln|\sqrt{e^2+1} + e| - \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}+1)\right]$$

10.1

7\. a) $1 - \frac{2}{n}$ is increasing since $\frac{2}{n}$ is decreasing.

or $f'(n) = 2/n^2 > 0$

b) $a_1 = 1$ $a_2 = 11$

$a_3 = 5 + \frac{6}{11} = \frac{61}{11} < 6$ $a_4 = 5 + \frac{66}{61} = \frac{371}{61} > 11$

$a_5 = 5 + \frac{6}{\text{"}>11\text{"}} < a_3$

After a_1, the evens increase which makes the odds decrease. Not monotone

7c) $a_1 = 7 \quad a_{n+1} = 7 - \frac{6}{a_n}$

$a_2 = 7 - \frac{6}{7} = \frac{43}{7} = 6\frac{1}{7} \quad a_3 = 7 - \frac{42}{43}$ or $7 - \frac{6}{\text{"}<7\text{"}} < 7 - \frac{6}{7}$

Induction:

$a_2 < a_1$ is true.

If $a_{n-1} < a_n$ then $a_{n+1} = 7 - \frac{6}{a_n} < 7 - \frac{6}{a_{n-1}} = a_n$.

because $0 < a_{n-1} < a_n$ iff $\frac{1}{a_{n-1}} > \frac{1}{a_n}$ iff $-\frac{1}{a_{n-1}} < -\frac{1}{a_n}$

iff $7 - \frac{6}{a_{n-1}} < 7 - \frac{6}{a_n}$

$\{a_n\}$ is decreasing.

8. a) $\left\{e^{\frac{n^2+2n}{3n^2+4n}}\right\} \quad \lim\limits_{n\to\infty} \frac{n^2+2n}{3n^2+4n} = \lim\limits_{n\to\infty} \frac{1+2/n}{3+4/n} = 1/3$

so $\lim\limits_{n\to\infty} e^{\frac{n^2+2n}{3n^2+4n}} = e^{1/3}$

b) $\ln[(1-\frac{2}{n})^n] = n\ln(1-2/n)$ Indeterminate form "$\infty \cdot 0$."

Put into a form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$" to use L'Hospital's rule. Use $\frac{\ln(1-2/n)}{1/n} \sim$ "$\frac{0}{0}$"

$$\frac{\ln(1-2/n)}{1/n} = \frac{\ln(n-2) - \ln n}{1/n}$$

$$\sim \frac{\frac{1}{n-2} - \frac{1}{n}}{-1/n^2} = \frac{n-(n-2)}{(n-2)n}(-n^2)$$

$$= \frac{-2n^2}{n^2-2n} = \frac{-2}{1-2/n} \xrightarrow[n\to\infty]{} -2$$

$$\ln a_n \xrightarrow[n\to\infty]{} -2 \quad \text{so} \quad a_n \xrightarrow[n\to\infty]{} e^{-2}$$

8c) $n(\tan\frac{1}{n})\tan\frac{1}{n} = a_n$

$$\lim_{n\to\infty} n\tan\frac{1}{n} = \lim_{n\to\infty}\frac{\tan 1/n}{1/n} = \lim_{x\to 0}\frac{\tan x}{x} = 1.$$

use L'Hoppital's Ru

Then $n\tan\frac{1}{n}\tan\frac{1}{n} \xrightarrow[n\to\infty]{} 1\cdot 0 = 0$ since $\tan 0 = 0$ and $\tan x$ is contu

9. a) $S_n = \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{11}} - \frac{1}{\sqrt{13}} + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{14}} + \ldots$

$+ \frac{1}{\sqrt{n-2}} - \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}}$

$= \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{11}} - \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$

$\lim_{n\to\infty} S_n = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{11}}$

b) $\sum_{n=1}^{\infty}\left(\ln\frac{1}{n} - \ln\frac{1}{n+1}\right)$ Note: $= \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

$= \overset{0}{\ln 1} - \ln\frac{1}{2} + \ln\frac{1}{2} - \ln\frac{1}{3} + \ldots + \ln\frac{1}{n} - \ln\frac{1}{n+1} + \ldots$

$= \lim_{n\to\infty} -\ln\frac{1}{n+1} = \lim_{n\to\infty} \ln(n+1) = \infty$ Diverges

c) $\sum_{n=2}^{\infty}\left[\frac{1}{\ln n} - \frac{1}{\ln(n+2)}\right]$

$S_n = \frac{1}{\ln 2} - \frac{1}{\ln 4} + \frac{1}{\ln 3} - \frac{1}{\ln 5} + \frac{1}{\ln 4} - \frac{1}{\ln 6}$

$\ldots + \frac{1}{\ln(n-1)} - \frac{1}{\ln(n+1)} + \frac{1}{\ln n} - \frac{1}{\ln(n+2)}$

$= \frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln(n+1)} - \frac{1}{\ln n+2} \xrightarrow[n\to\infty]{} \boxed{\frac{1}{\ln 2} + \frac{1}{\ln 3}}$

d) $S_n = \cos 1 - \cos\frac{1}{2} + \cos\frac{1}{2} - \cos\frac{1}{3} + \dots + \cos\frac{1}{n} - \cos\frac{1}{n+1}$

$$= \cos 1 - \cos\frac{1}{n+1} \xrightarrow[n\to\infty]{} \cos 1 - \cos 0 = \boxed{(\cos 1) - 1}$$

e) $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$ $\quad a_n = n \sin\frac{1}{n} = \frac{1}{x}\sin x \quad x = \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Diverges by the Divergence Test.

f) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{3}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{3}{2} \frac{1}{1-\frac{1}{2}} = 3$

g) $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$ and $\sum_{n=1}^{\infty} \frac{2}{3^n}$ both converge so we can add their sums.

$$\sum_{n=1}^{\infty} \frac{4}{n(n+1)} = 4\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 4$$

telescoping

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1} = \frac{2}{3}\frac{1}{1-\frac{1}{3}} = 1$$

So the series $\sum_{n=1}^{\infty}\left(\frac{4}{n(n+1)} + \frac{2}{3^n}\right)$ converges to 5.

h) $\sum_{n=5}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=5}^{\infty} \left(-\frac{1}{3}\right)^n = \left(-\frac{1}{3}\right)^5 \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}$

$= \left(-\frac{1}{3}\right)^5 \frac{1}{1+1/3} = -\frac{3}{4} \cdot \frac{1}{3^5} = \frac{-1}{324}$

10. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\qquad \frac{n}{e^n} = ne^{-n}$ We can integrate xe^{-x} by parts. xe^{-x} is a decreasing fncn.

$$\int_1^{\infty} xe^{-x}\,dx = -xe^{-x}\Big|_1^{\infty} + \int_1^{\infty} e^{-x}\,dx$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{T\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_1^T$$

$$= \lim_{T\to\infty} \left(-Te^{-T} - e^{-T} - (-e^{-1} - e^{-1})\right)$$

$$= \frac{2}{e}$$

Since xe^{-x} is decreasing and the integral converge the series also converges.

$$(n+2)e^{-(n+1)} = \int_{n+1}^{\infty} xe^{-x}\,dx \le R_n \le \int_n^{\infty} xe^{-x}\,dx = (n+1)e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p} \quad p > 1$$

11a) $\frac{\ln x}{x^p}$ is a decreasing function. Int. by parts.

$$\int_1^T \frac{\ln x}{x^p}\,dx = \frac{x^{1-p}}{1-p}\ln x\Big|_1^T - \frac{1}{1-p}\int_1^T x^{-p}\,dx$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-p}\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{1-p}x^{1-p} \end{array}}$$

$$= \left[\frac{x^{1-p}}{1-p}\ln x - \frac{1}{(1-p)^2}x^{1-p}\right]\Bigg|_1^T$$

$$= \frac{T^{1-p}\ln T}{1-p} - \frac{1}{(1-p)^2}T^{1-p} - 0 + \frac{1}{(1-p)^2} \xrightarrow[T\to\infty]{} \frac{1}{(1-p)^2}$$

since $p > 1$.

Ex. If $p = 3$, $\frac{T^{-2}\ln T}{-2} - \frac{1}{(4)}T^{-2} + \frac{1}{4} \xrightarrow[T\to\infty]{} \frac{1}{4}$

The series also converges by the integral test.

b) $\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$, $\frac{1}{n-\ln n} > \frac{1}{n}$ diverges by comparison to $\sum \frac{1}{n}$ which diverges.

11c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$

$\frac{1}{n+\ln n} < \frac{1}{n}$ but $\sum \frac{1}{n}$ diverges so comparison does not work. We use the limit comparison:

$$\frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = \frac{n}{n+\ln n} = \frac{1}{1+\frac{\ln n}{n}} \xrightarrow[n\to\infty]{} 1$$

so both series diverge since $\sum \frac{1}{n}$ does.

11d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2\ln n}$ $\quad \sum \frac{1}{n^2}$ converges and $\frac{1}{n^2+2\ln n} \leq \frac{1}{n^2}$ so the series converges by Comparison.

11e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2-3\ln n}$ $\quad \sum \frac{\ln n}{n^2}$ converges by 11a but $\frac{\ln n}{n^2-3\ln n} \geq \frac{\ln n}{n^2}$ so we use the limit comparison test.

$$\frac{\frac{\ln n}{n^2-3\ln n}}{\frac{\ln n}{n^2}} = \frac{n^2}{n^2-3\ln n} = \frac{1}{1-\frac{3\ln n}{n^2}} \xrightarrow[n\to\infty]{} 1$$

so both series converge since $\sum \frac{\ln n}{n^2}$ does.

11f) $\displaystyle\sum_{n=2}^{\infty} \frac{-1+2^n}{n \cdot 2^n \ln n}$

$$\frac{-1+2^n}{n \cdot 2^n \ln n} = \frac{-1}{n \cdot 2^n \ln n} + \frac{1}{n \ln n} = -a_n + b_n .$$

$\sum -a_n$ converges by comparison to $-\sum \left(\frac{1}{2}\right)^n$ but

$\sum b_n$ diverges by the integral test:

$$\int_2^T \frac{1}{x \ln x}\, dx = \ln(\ln x) \Big|_2^T \xrightarrow[T \to \infty]{} \infty$$

so $\sum(-a_n + b_n)$ must diverge.