Math 152 Exam 3 Review Solutions
Fall '09

Recall $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or use L'Hospital's Rule.

1a) Since $\frac{1}{n} \to 0$ as $n \to \infty$,

$$n \sin \frac{1}{n} = \frac{\sin \frac{1}{n}}{\frac{1}{n}} \to 1 \quad n \to \infty$$

Diverges by Divergence test.

1b) $\frac{3}{2^n} = \frac{3}{2} \cdot \left(\frac{1}{2}\right)^{n-1}$ $\quad a = \frac{3}{2} \quad r = \frac{1}{2}$

Geometric series $\sum_{n=1}^{\infty} \frac{3}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{3}{2} \cdot \frac{1}{1-\frac{1}{2}} = 3$

1c) Since $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{4}{n^2}$; and $\sum_{n=1}^{\infty} \frac{4}{5^n} = \frac{4}{5} \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1}$ converges also, we can split the two series.

$$\sum_{n=1}^{N} \frac{4}{n(n+1)} = \sum_{n=1}^{N} 4\left(\frac{1}{n} - \frac{1}{n+1}\right) = 4\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{N} - \frac{1}{N+1}\right)$$

$$= 4\left(1 - \frac{1}{N+1}\right) \qquad \sum_{n=1}^{\infty} \frac{4}{n(n+1)} = 4 \lim_{N \to \infty} \left(1 - \frac{1}{N+1}\right) = 4$$

$$\frac{4}{5} \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1} = \frac{4}{5} \cdot \frac{1}{1-\frac{1}{5}} = 1$$

Then the original sum is $4 + 1 = 5$

2. a) $\sum_{n=1}^{\infty} \frac{n}{e^n} = \sum_{n=1}^{\infty} f(n)$ where $f(x) = \frac{x}{e^x} = xe^{-x}$

f is decreasing and non-negative so we can use the integral test. Integrating by parts:

$$\int_1^N xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$$

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}\,dx$

$$= -xe^{-x} - e^{-x} \Big|_1^N$$

$$= -Ne^{-N} - e^{-N} + 2e^{-1} \underset{N\to\infty}{\longrightarrow} 2e^{-1}$$

So the series converges also and for

$R_N = \sum_{k=N+1}^{\infty} \frac{k}{e^k}$

$$R_N \leq \int_N^\infty xe^{-x}\,dx$$

$$= -xe^{-x} - e^{x} \Big|_N^\infty$$

$$= +Ne^{-N} + e^{-N} = (N+1)e^{-N}$$

Also:

$$R_N \geq \int_{N+1}^{\infty} xe^{-x}\,dx$$

$$= (N+2)\,e^{-(N+1)}$$

2b) Using the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, e^{-(n+1)}}{n!\, e^{-n}} = (n+1)e^{-1} \xrightarrow[n\to\infty]{} \infty$$

The series diverges.

2c) We can use the integral test or the ratio test to show the series converges. Here the ratio test is used:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 r^{n+1}}{n^2 r^n}\right| = \left(1+\frac{1}{n}\right)^2 |r| \xrightarrow[n\to\infty]{} |r|$$

The series converges since $|r| < 1$.

2d) Using the ratio test:

$$\frac{\dfrac{3\cdot5\cdot7\cdot\ldots\cdot(2n+3)}{(n+1)!\ 2^{n+1}}}{\dfrac{3\cdot5\cdot7\ldots\ (2n+1)}{n!\ \ 2^n}} = \frac{(2n+3)(2n+2)}{n+1}\cdot\frac{1}{2}$$

$$= 2n+3 \xrightarrow[n\to\infty]{} \infty$$

Diverges.

2e) By the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \frac{e}{n+1} \to 0 \text{ Conv,}$$

The sum of this series is e^e.

3. a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ Converges by the

alternating series test and since $\frac{1}{\ln n}$ is decreasing to 0 as $n \to \infty$.

$$|R_n| \leq \frac{1}{\ln(n+1)}$$

Does it converge absolutely? No by comparison to $\sum_{n=2}^{\infty} \frac{1}{n}$ which diverges.

$$\frac{1}{\ln n} > \frac{1}{n} \quad \text{since } \ln n < n$$

3 b) $\lim_{x\to 0^+} \tan x = \tan 0 = 0$ so with $x = \frac{1}{n}$,

$\lim_{n\to\infty} \tan\frac{1}{n} = 0$. $\tan x$ is increasing

near 0 as x increases. So $\tan\frac{1}{n}$ is decreasing as $\frac{1}{n}$ decreases, as $n \to \infty$

The series converges by the alternating series test.

Does it converge Absolutely? No

We know $\sum \frac{1}{n}$ diverges and

$$\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{1/n} = \lim_{x\to 0} \frac{\tan x}{x} = 1$$ so $\sum \tan\frac{1}{n}$

also diverges by the Limit Comparison Test.

3c) $$\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$ which is decreasing

to 0 as $n \to \infty$.

$\sum_{n=1}^{\infty} (-1)^n \left[\sqrt{n+1} - \sqrt{n}\right]$ converges by the

Alternating Series test, but does

not converge absolutely since

$\frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{2(\sqrt{n+1})}$ and $\sum \frac{1}{\sqrt{n+1}}$ diverges.

4. $f(x) = \arctan x = \int \frac{1}{1+x^2}\,dx$

$$\frac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n \quad \text{Substitute } u = x^2:$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{Find the anti-derivative:}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \quad \text{Subst. } x=0:$$

$$0 = 0 + C \quad \text{so } C = 0.$$

$I = [-1, 1] \quad R = 1$

5. $g(x) = x \arctan x^2$

Substitute x^2 into $\arctan u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}$

then multiply by x.

$$x \arctan x^2 = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2n+1} \quad I = [-1, 1] \quad R = 1$$

6. $\ln(4+x) = \int \frac{1}{4+x}\,dx = \int \frac{1}{4(1+\frac{x}{4})}\,dx$

$$\frac{1}{4+x} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^{n+1}}$$

$$\ln(4+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)4^{n+1}} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n 4^n} + C$$

Substitute $x=0$ to find $\ln 4 = C \quad I = (-4, 4] \quad R = 4$

$$\ln(4+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n\,4^n} + \ln 4$$

$I = (-4, 4]$ $R = 4$

7. $\ln(4+x^2)$: Substitute x^2 into $\ln(4+u)$:

$$\ln(4+x^2) = \ln 4 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n\,4^n}$$

$I = [-2, 2]$ $R = 2$

8. $f(x) = \frac{x}{(1-x)^2} = x \cdot \frac{d}{dx}\left(\frac{1}{(1-x)}\right)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{so} \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

Then $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$ $I = (-1, 1)$ $R = 1$

9. $\frac{1}{(5-2x)^2} = \frac{1}{5(1-\frac{2x}{5})^2} = \frac{1}{5(1-u)^2}$ if $u = \frac{2x}{5}$

$$\frac{1}{(1-u)^2} = \frac{d}{du}\left(\frac{1}{1-u}\right) = \frac{d}{du}\left(\sum_{n=0}^{\infty} u^n\right)$$

$= \sum_{n=1}^{\infty} n u^{n-1}$ Substitute $u = \frac{2x}{5}$ and

divide by 5 : $\frac{1}{5}\sum_{n=1}^{\infty} n\left(\frac{2x}{5}\right)^{n-1}$

$$= \sum_{n=1}^{\infty} n \frac{2^{n-1}}{5^n} x^{n-1} = \sum_{n=0}^{\infty} (n+1) \frac{2^n x^n}{5^{n+1}}$$

$I = (-5/2, 5/2)$ $R = 5/2$

10. $f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = \frac{-1}{x^2} \qquad f''(2) = \frac{-1}{2^2}$

$f'''(x) = \frac{(-1)(-2)}{x^3} \qquad f'''(2) = \frac{(-1)^2 \cdot 2 \cdot 1}{2^3}$

$f^{iv}(x) = \frac{(-1)(-2)(-3)}{x^4} \qquad f^{iv}(2) = \frac{(-1)^3 \cdot 3 \cdot 2 \cdot 1}{2^4}$

$\vdots \qquad\qquad \vdots$

$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{n} \qquad f^n(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$

$$c_n = \frac{(-1)^{n-1}(n-1)!}{n!\, 2^n} = \frac{(-1)^{n-1}}{n\, 2^n} \qquad n \geq 1$$

$$f(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 2^n}(x-2)^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \cdot \frac{1}{2} |x-2| \qquad R = 2$$

$I = (0, 4]$ If $x=0$ the series is

$$\sum \frac{1}{n}(-1)^{2n-1} = -\sum \frac{1}{n} \text{ diverges.}$$

If $x=4$, it is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ which converges by the alternating Series Test.

11. $f(x)=\sqrt{x}$

$$f'(x)=\frac{1}{2}x^{-1/2}$$

$$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$$

$$f^{IV}(x)=\frac{3\cdot5(-1)^3}{2^4}x^{-7/2}$$

$$f^{n}(x)=\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-3)(-1)^{n-1}}{2^n}x^{-\frac{2n-1}{2}}$$

$$c_n=\frac{1\cdot3\cdot5\cdot\ldots(2n-3)(-1)^{n-1}}{n!\,2^n}$$

$$f(x)=1+\frac{1}{2}(x-1)+\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-3)(-1)^{n-1}}{n!\,2^n}(x-1)^n$$

$$\left|\frac{c_{n+1}}{c_n}\right|=\frac{2n-3}{n+1}\cdot\frac{1}{2}\to1\ (n\to\infty)\quad\text{so } R=1$$

$$I=[0,2]$$

We can show it converges at $x=0$:

$$a_n = \frac{1\cdot 3\cdot 5\cdot \ldots [2(n-1)-1]}{2\cdot 4\cdot 6\cdot \ldots 2(n-1)} \cdot \frac{1}{2n}(-1)^n = b_n(-1)^n$$

where $b_n < \frac{1}{2n}$ and is decreasing to 0

At $x=2$, we need a test not covered in this course.

12. a) $f(x) = \cos x$ $\quad f(0)=1 \quad f^{2n}(0) = (-1)^n$

$f'(x) = -\sin x \qquad f^{2n+1}(0) = 0$

$f''(x) = -\cos x$

$f^3(x) = \sin x$

repeat

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

b) $f(x) = \sin x$ $\quad f(0)=0 \quad f^{2n}(0) = 0$

$f'(x) = \cos x \qquad f^{2n+1}(0) = (-1)^n$

$f''(x) = -\sin x$

$f^3(x) = -\cos x$

repeat

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

c) $f^n(x) = e^x$

$f^n(0) = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

All 3 series converge on $(-\infty, \infty)$.

13. $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

14. $$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$ Substitute $u = -x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Multiply by x:

$$xe^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!}$$

13. $g(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$f(x) = \frac{g(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

$x \neq 0$

14. $e^{-x^2} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \qquad u = -x^2$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$x e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$

15. $f(x) = \sin x \qquad \sin \pi/6 = 1/2$

$f'(x) = \cos x \qquad \cos \pi/6 = \sqrt{3}/2$

$f''(x) = -\sin x \qquad -\sin \pi/6 = -1/2$

$f^3(x) = -\cos x \qquad -\cos \pi/6 = -\frac{\sqrt{3}}{2}$

For the error: $f^4(x) = \sin x$

$T_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2$

$- \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$

$I = [0, \frac{\pi}{3}]$

$$|f(x) - T_3(x)| \leq \max_I \frac{|f^4(x)|\ |x - \frac{\pi}{6}|^4}{4!}$$

$$\max_{I=[0,\pi/3]} |\sin x| = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and } |x - \frac{\pi}{6}| \leq \frac{\pi}{6}$$

$$|f(x) - T_3(x)| \leq \frac{\sqrt{3}}{48}\left(\frac{\pi}{6}\right)^4$$

16. $f(x) = \sqrt{1-x}$ $\qquad f(0) = 1 \quad c_0 =$

$f'(x) = -\frac{1}{2}(1-x)^{-1/2}$ $\qquad f'(0) = -1/2 \quad c_1 =$

$f''(x) = -\frac{1}{4}(1-x)^{-3/2}$ $\qquad f''(0) = -1/4 \quad c_2 =$

$f^3(x) = -\frac{3}{8}(1-x)^{-5/2}$

$$T_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$|f(x) - T_2(x)| \leq \max_{|x|<1/4} \frac{3}{8}(1-x)^{-5/2}\ \frac{1}{!} \cdot |x|$$

$\max|-x|^{-5/2}$ occurs at $x = 1/4$ where $1-x$ is it smallest value, $3/4$.

$$|f(x) - T_2(x)| \leq \frac{3}{8}\left(\frac{3}{4}\right)^{-5/2} \cdot \frac{1}{6} \cdot \left(\frac{1}{4}\right)^3$$

$$= \frac{1}{(32)(3^{5/2})} = \frac{1}{288\sqrt{3}}$$

17. $\sqrt{1-x^2} = \sqrt{1-u}$ where $u = x^2$.

From #16

$T_2(u) = 1 - \frac{1}{2}u - \frac{1}{8}u^2$
for $\sqrt{1-u}$

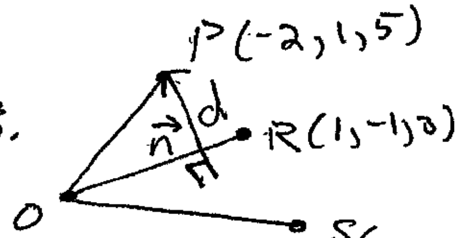
Substituting x^2 for u:

$T_4(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4$
for $\sqrt{1-x^2}$

$|u| \le 1/4 \iff |x^2| \le \frac{1}{4} \iff |x| \le 1/2$

So $\left|\sqrt{1-x^2} - T_4(x)\right| \le \frac{1}{288\sqrt{3}}$ from probl 16.

18.



Find $\vec{n}$ orthogonal to $\overrightarrow{OR}$ and $\overrightarrow{OS}$.

Find $\left|\text{comp}_{\vec{n}} \overrightarrow{OP}\right|$.

$\vec{n} = \overrightarrow{OR} \times \overrightarrow{OS}$

$$\text{comp}_{\vec{n}} \overrightarrow{OP} = \frac{\overrightarrow{OP} \cdot (\overrightarrow{OR} \times \overrightarrow{OS})}{|\overrightarrow{OR} \times \overrightarrow{OS}|}$$

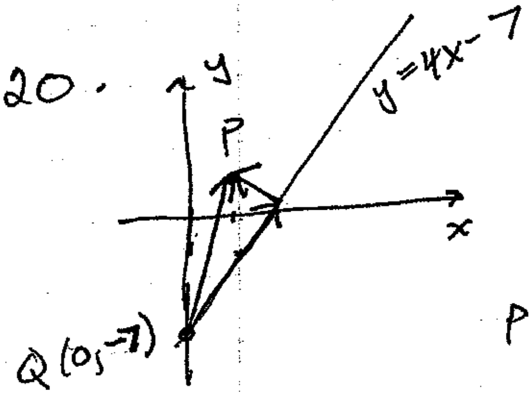
$$= \frac{\begin{vmatrix} -2 & 1 & 5 \\ 1 & -1 & 3 \\ 2 & -1 & 6 \end{vmatrix}}{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 2 & -1 & 6 \end{vmatrix}} = \frac{-2(-6+3) - 1(6-6) + 5(-1+2)}{|\vec{i}(-6+3) - \vec{j}(6-6) + \vec{k}(-1+2)|}$$

$$= \frac{6+5}{\sqrt{9+1}} = \frac{11}{\sqrt{10}}$$

19. $\begin{vmatrix} -2 & 1 & 5 \\ 1 & -1 & 3 \\ 2 & -1 & 6 \end{vmatrix}$

$= -2(-6+3) - 1(6-6) + 5(-1+2)$

$= 6 + 5 = 11$ which we knew from #18.

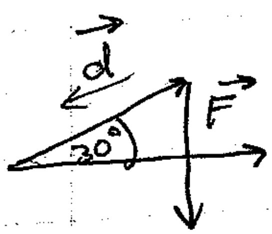
20. 

$Q(0,-7)$ and $R(1,-3)$ lie on the line $y = 4x - 7$. $\vec{v} = \overrightarrow{QR} = \langle 1, 4 \rangle$ is parallel to the line.

$\vec{v}^{\perp} = \langle -4, 1 \rangle$ is $\perp$ to the line. Find $\text{comp}_{\vec{v}^{\perp}} \overrightarrow{QP}$.

$\overrightarrow{QP} = \langle 1, 9 \rangle \qquad \overrightarrow{QP} \cdot \dfrac{\vec{v}^{\perp}}{|\vec{v}^{\perp}|} =$

$= \dfrac{\langle 1, 9 \rangle \cdot \langle -4, 1 \rangle}{\sqrt{17}} = \dfrac{-4+9}{\sqrt{17}} = \dfrac{5}{\sqrt{17}}$

21. 

$\vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos(60°)$

$= 15(2) \cdot \frac{1}{2} = 15$ foot-lbs.