Math 251 Section 13.5       Double Integrals in Polar Coordinates

A basic region in polar form is \( \theta_1 \leq \theta \leq \theta_2, \quad r_1 \leq r \leq r_2 \) which describes a sector of an annulus in the \( x,y \)-plane. Its area is
\[
\frac{1}{2} (\theta_2 - \theta_1)(r_2^2 - r_1^2) = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr \, d\theta.
\]

To convert an integral from Cartesian to polar coordinates:
\[
\int \int_A f(x, y) \, dA = \int \int_A f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]
We use
\[x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}\]
This means that \( r \) is always nonnegative.

Examples:

1. \( R \) is the region in the first octant outside \( x^2 + y^2 = 2x \) and inside \( x^2 + y^2 = 4 \). Graph \( R \) and describe it in polar coordinates.

Evaluate \( \int \int_R y \, dA \).

2. Find the volume of the solid bounded by the paraboloid \( z = 16 - 3x^2 - 3y^2 \) and the plane \( z = 4 \).

3. #20 in Stewart Find the volume between the paraboloid \( z = 3x^2 + 3y^2 \) and \( z = 4 - x^2 - y^2 \).

4. #28 in Stewart Convert the integral \( \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \) to polar coordinates.