14.3 The Fundamental Theorem for Line Integrals \[ \text{FTLI} \]

Assumptions:
1) $C: x = x(t), y = y(t), a \leq t \leq b$ is a smooth curve, i.e. $x'(t)$ and $y'(t)$ are continuous.

2) $\overrightarrow{F}(x,y)$ is continuous on the domain $D$ containing $C$, i.e. $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous.

\[ \text{FTLI: Theorem} \quad \int_{C} \overrightarrow{F} \cdot d\overrightarrow{r} = F(x(b), y(b)) - F(x(a), y(a)) \]

This follows from the Fundamental Theorem of Calculus since:

\[ \int_{C} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{a}^{b} \left( \frac{\partial F}{\partial x} x'(t) + \frac{\partial F}{\partial y} y'(t) \right) dt \]

\[ = \int_{a}^{b} \frac{d}{dt} \left[ F(x(t), y(t)) \right] dt = F(x(b), y(b)) - F(x(a), y(a)) \]

Definition: \[ \int_{C} \overrightarrow{F} \cdot d\overrightarrow{r} \text{ is independent of path} \]

for any two paths $C_1$ and $C_2$ in $D$ with the same initial and terminal pts., \[ \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r} \]
Assume \( F \) is continuous on an open, connected region \( D \). \( D \) is connected if any two pts in \( D \) can be joined by a path in \( D \).

**Theorem:** \( F \) is independent of path in \( D \) if and only if \( F \) is conservative.

\[
\nabla \Phi = F
\]

**Example:**

Show \( F = (x^2+y)i + (y^2+x)j \) is conservative by finding a potential function.

**Solution:** We need \( \frac{\partial f}{\partial x} = x^2 + y \) and \( \frac{\partial f}{\partial y} = y^2 + x \).

By (1): \( f = \frac{1}{3} x^3 + xy + g(y) \)

By (2): \( f = \frac{1}{2} y^3 + xy + h(x) \)

Thus \( f(x, y) = \frac{1}{3} x^3 + \frac{1}{3} y^3 + xy \)

* Assume \( D = \text{domain of } F \) is open and simply connected.

**Theorem**

A vector field \( F(x, y) = P(x, y)i + Q(x, y)j \) is conservative if and only if

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
\]

* \( D \) is open if it contains no boundary points, it must contain a ball of positive radius centered at \((a, b)\) for each \((a, b) \in D\).

* \( D \) is simply connected if it has no holes.
Examples

Determine whether \( \mathbf{F} \) is conservative. If so, find \( f \).

#6 in Stewart

\[
\mathbf{F}(x, y) = (y \cos x - \cos y) \mathbf{i} + (\sin x + x \sin y) \mathbf{j}
\]

\[
P(x, y) = y \cos x - \cos y \quad \frac{\partial P}{\partial y} = \cos x + \sin y
\]

\[
Q(x, y) = \sin x + x \sin y \quad \frac{\partial Q}{\partial x} = \cos x + \sin y
\]

Find \( f \):

\[
\frac{\partial f}{\partial x} = y \cos x - \cos y
\]

so

\[
f(x, y) = y \sin x - x \cos y + g(y)
\]

\[
\frac{\partial f}{\partial y} = \sin x + x \sin y
\]

\[
f(x, y) = y \sin x - x \cos y + h(x)
\]

so

\[
f(x, y) = y \sin x - x \cos y
g(y) = 0 = h(x)
\]

#8

\[
\mathbf{F}(x, y) = (y e^{xy} + 4x^2y) \mathbf{i} + (x e^{xy} + x^2) \mathbf{j}
\]

#3

\[
\mathbf{F}(x, y) = (x^2 + y) \mathbf{i} + x^2 \mathbf{j}
\]

12. Evaluate \( \int \mathbf{F} \cdot d\mathbf{r} \) by finding a potential function \( f \) for \( \mathbf{F} \) and using \( f \) to evaluate the integral.

\[
\mathbf{F}(x, y) = y \mathbf{i} + x \mathbf{j}
\]

\( C: y = x^4 - x^3 \) from (1, 0) to (2, 8)

\#33. \( \mathbf{F}(x, y) = \frac{-y \mathbf{i} + x \mathbf{j}}{x^2 + y^2} \quad C: \text{unit circle} \quad x^2 + y^2 = 1 \)

\( D \) is not simply connected
ence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the Law of Conservation of Energy and it is the reason the vector field is called conservative.

**EXERCISES 14.3**

(a) Determine whether or not \( F \) is a conservative vector field. If it is, find a function \( f \) such that \( F = \nabla f \).

1. \( F(x, y) = (2x - 3y)i + (3y - 2x)j \)
2. \( F(x, y) = (3x^2 - 4y)i + (4y^2 - 2x)j \)
3. \( F(x, y) = (x^3 + y)i + x^2j \)
4. \( F(x, y) = (x^2 + y)i + (y^2 + x)j \)
5. \( F(x, y) = (1 + 4x^3y)i + 3x^2yj \)
6. \( F(x, y) = (\cos x - \cos y)i + (\sin x + \sin y)j \)
7. \( F(x, y) = (e^{xy} + xy)i + 2x^2yj \)
8. \( F(x, y) = (y^2x - 4x^2y)i + (xy^2 + x^3)j \)
9. \( F(x, y) = (ye^{x} + \sin y)i + (e^x + x\cos y)j \)
10. \( F(x, y) = (x + y^2)i + (2xy + y^2)j \)

(b) Find a function \( f \) such that \( F = \nabla f \) along the given curve \( C \).

11. \( F(x, y) = xi + yj \),
   \( C \) is the arc of the parabola \( y = x^2 \) from \((-1,1)\) to \((3,9)\).
12. \( F(x, y) = yi + xj \),
   \( C \) is the arc of the curve \( y = x^3 - x^4 \) from \((1,0)\) to \((2,8)\).
13. \( F(x, y) = 2xyi + 3x^2y^2j \),
   \( C \) is the line segment from \((2,1,4)\) to \((8,3,-1)\).
14. \( F(x, y, z) = 2xy^2i + 3x^2y^2j + 4x^2y^2k \),
   \( C \) is the line segment from \((1,1,1)\) to \((3,2,-1)\).
15. \( F(x, y, z) = 4xyi + 2x^2j + 3x^2k \),
   \( C \) is the line segment from \((1,2,3)\) to \((2,1,4)\).

\[ \int_C 2x \sin y \, dx + (x^2 \cos y - 3y^2) \, dy, \]
\( C \) is any path from \((-1,0)\) to \((5,1)\).

\[ \int_C (2x^2 - 12x^3) \, dx + (4xy - 9x^2y^2) \, dy, \]
\( C \) is any path from \((1,1)\) to \((3,2)\).

21–22 **Find the work done by the force field \( F \) in moving an object from \( P \) to \( Q \).**

21. \( F(x, y) = x^3y^3i + x^3y^3j \); \( P(0,0), Q(2,1) \)
22. \( F(x, y) = (y^2x^2)i - (2y^2)j \); \( P(1,1), Q(4,2) \)

\[ \text{Is the vector field shown in the figure conservative? Explain.} \]

23. From a plot of \( F \) guess whether it is conservative. Then determine whether your guess is correct.

24. \( F(x, y) = (2xy + \sin y)i + (x^2 + x\cos y)j \)
25. \( F(x, y) = \frac{(x - 2y)}{\sqrt{1 + x^2 + y^2}}i \)
26. Let \( F = \nabla f \), where \( f(x, y) = \sin(x - 2y) \). Find curves \( C_1 \) and \( C_2 \) that are not closed and satisfy the equation.

\[ \int_{C_1} F \cdot dr = 0 \quad \text{and} \quad \int_{C_2} F \cdot dr = 1 \]

27. **Show that if the vector field \( F = Pi + Qj + RK \) is conservative and \( P, Q, R \) have continuous first-order partial derivatives, then**

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial x} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y} \]

28. **Use Exercise 27 to show that the line integral**

\[ \int_C y \, dx + x \, dy + xy \, dz \]

**is not independent of path.**

29–32 **Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.**

29. \( \{(x, y) \mid x > 0, y > 0\} \)
30. \( \{(x, y) \mid x \neq 0\} \)
31. \( \{(x, y) \mid 1 < x^2 + y^2 < 4\} \)
CONSERVATION OF ENERGY

Let's apply the ideas of this chapter to a continuous force field $F$ that moves an object along a path $C$ given by $r(t)$, $a \leq t \leq b$, where $r(a) = A$ is the initial point and $r(b) = B$ is the terminal point of $C$. According to Newton's Second Law of Motion (see Section 11.8), the force $F(r(t))$ at a point on $C$ is related to the acceleration $a(t) = r''(t)$ by the equation

$$F(r(t)) = ma''(t)$$

So the work done by the force on the object is

$$W = \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt$$

$$= \int_a^b ma''(t) \cdot r'(t) \, dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} [r'(t) \cdot r'(t)] \, dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |r'(t)|^2 \, dt$$

$$= \frac{m}{2} \int_a^b \left( |r'(t)|^2 - |r'(a)|^2 \right) \, dt$$

Therefore

$$W = \frac{1}{2} m |v(b)|^2 - \frac{1}{2} m |v(a)|^2$$

where $v = r'$ is the velocity.

The quantity $\frac{1}{2} m |v(t)|^2$, that is, half the mass times the square of the speed, is called the kinetic energy of the object. Therefore, we can rewrite Equation 15 as

$$W = K(B) - K(A)$$

which says that the work done by the force field along $C$ is equal to the change in kinetic energy at the endpoints of $C$.

Now let us further assume that $F$ is a conservative force field; that is, we can write $F = \nabla f$. In physics, the potential energy of an object at the point $(x, y, z)$ is defined as $P(x, y, z) = -f(x, y, z)$, so we have $F = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C F \cdot dr = -\int_C \nabla P \cdot dr$$

$$= -[P(r(b)) - P(r(a))]$$

$$= P(A) - P(B)$$

Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point $A$ to another point $B$ under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the Law of Conservation of Energy and it is the reason the vector field is called conservative.