Math 251  Section 14.6  Parametric Surfaces and Their Areas

A surface in $\mathbb{R}^3$ has a 3-component vector equation which depends on two independent parameters.

$$\vec{r}(u,v) = x(u,v) \hat{i} + y(u,v) \hat{j} + z(u,v) \hat{k} \quad (u,v) \text{ in } D.$$ 

Example 1: The surface of the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 9$, has vector equation

$$\vec{r}(x,y) = x \hat{i} + y \hat{j} + (x^2 + y^2) \hat{k} \quad D = \{(x,y) : \quad x^2 + y^2 \leq 9\}$$

The parameter domain is an important part of the definition of the surface.

Example 2: Parameterize the surface of the cylinder, $x^2 + y^2 = 4$ between the planes $z = 0$ and $x + z = 3$.

A point on this surface lies on the circle of radius 2 centered at $(0,0,z)$.

Let the parameters be $\theta$ and $z$. Then $x = 2 \cos \theta$, $y = 2 \sin \theta$, $z = z$

$$\vec{r}(\theta,z) = 2 \cos \theta \hat{i} + 2 \sin \theta \hat{j} + z \hat{k} \quad \text{for} \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3 - 2 \cos \theta$$

Example 3: Parameterize the surface of the plane $x + z = 3$ over the disk $x^2 + y^2 = 4$, $z = 0$. 
Tangent Planes to Surfaces: Let the surface, \( S \), be given by
\[
\vec{r}(u,v) = x(u,v) \vec{i} + y(u,v) \vec{j} + z(u,v) \vec{k} \quad (u,v) \text{ in } D \quad \text{and } P(a,b,c) \text{ a point on } S.
\]
Let \( \vec{r}(u_0,v_0) = a \vec{i} + b \vec{j} + c \vec{k} \).

The vectors \( \frac{\partial \vec{r}}{\partial u}(u_0,v_0), \frac{\partial \vec{r}}{\partial v}(u_0,v_0) \) are both tangent to \( S \) at \( P(a,b,c) \).

So their cross product is a normal vector to the tangent plane to \( S \) at \( P(a,b,c) \).

Example 2: Find the tangent plane to the surface \( z = 4 - x^2 - y^2 \), \( z \geq 0 \) at \( P(1,-1,2) \).

Example 3: Find the tangent plane to the surface parameterized by
\[
\vec{r}(u,v) = u^2 \vec{i} + u e^v \vec{j} + (v - u) e^v \vec{k} \quad \text{for } |u| \leq 2, \quad |v| \leq 2 \quad \text{at the point } P(1,-1,1).
\]

If the surface is given by \( z = f(x,y) \) then \( \vec{r}(x,y) = x \vec{i} + y \vec{j} + f(x,y) \vec{k} \) and as in Chapter 12, the normal is \( -\frac{\partial f}{\partial x} \vec{i} - \frac{\partial f}{\partial y} \vec{j} + \vec{k} = \vec{r}_x \times \vec{r}_y \) evaluated at the given point.
Surface Areas:

Definition: A surface given by \( \mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \) \((u, v) \text{ in } D\)

is called smooth if \( \mathbf{r}_u \times \mathbf{r}_v \) is never zero in \( D \). For such a surface, the surface area is

\[
\text{surface area} = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA,
\]

(assuming the surface is covered exactly once on the parameter domain.)

In the case that the surface is given by \( \mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k} \) then

\[
\text{surface area} = \iint_D \sqrt{\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + 1} \, dA
\]

Example 4: Find the surface area of the part of the paraboloid \( x = y^2 + z^2 \) that lies inside the cylinder \( y^2 + z^2 = 9 \).