Math 251 Section 14.7 Surface Integrals and Flux

**Surface Integrals** For \( f(x, y, z) \) defined on a surface, \( S \), given by
\[
\vec{r}(u, v) = x(u, v) \vec{i} + y(u, v) \vec{j} + z(u, v) \vec{k}, \quad (u, v) \text{ in } D,
\]
the surface integral of \( f(x, y, z) \) over the surface is
\[
\iint_S f(x, y, z) dS = \iint_D f(x, y, z) \left| \vec{r}_u \times \vec{r}_v \right| dA.
\]

Examples:

#4 pg 92 in Stewart Evaluate
\[
\iint_S xz \ dS
\]
where \( S \) is the triangle with vertices (1,0,0), (0,1,0) and (0,0,1).

#6 in Stewart Evaluate
\[
\iint_S (y^2 + z^2) \ dS
\]
where \( S \) is the part of the paraboloid \( x = 4 - y^2 - z^2 \) in front of the \( y,z \)-plane.

**Flux** Introduction to Flux:

Consider a skewed cylindrical tube and a fluid flowing through the tube at speed \( v \). (such as blood flowing through a section of an artery) In \( \Delta t \) seconds the fluid flows \( v \Delta t \) units along the slant height of the tube. Let the top of the cylinder be the incremental surface area, \( \Delta S \). **What volume of fluid flows across \( \Delta S \) in \( \Delta t \) seconds?** It is the volume of the skewed cylinder.

The volume of the skewed cylinder is the vertical height times the area of the base = \( v \Delta t \cos \theta \Delta S = \vec{v} \cdot \vec{n} \Delta S \Delta t \)

\[
\vec{v} \cdot \vec{n} \Delta S \Delta t
\]

Here \( \vec{n} \) is the chosen unit normal to the base and \( \vec{v} \) is in the direction of the slant height. Note \( \Delta S \) has a chosen orientation.

If we let \( \Delta t = 1 \) we have the volume of fluid that flows across \( \Delta S \) each second.

For a general surface, we add and take the limit to get the volume per second that flows across the surface.
Flux Definition  For a continuous vector function, \( \vec{F}(x, y, z) \), and an oriented surface, \( S \), the flux of \( \vec{F} \) across \( S \) is \( \iint_S \vec{F} \cdot \vec{n} \, dS \) where \( \vec{n} \) is the unit normal to \( S \). This is also written as

\[
\iint_S \vec{F} \cdot dS.
\]

To evaluate this, first parameterize the surface as \( \vec{r}(u, v), (u, v) \in D \). Then

\[
\vec{n} \, dS = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA = \pm (\vec{r}_u \times \vec{r}_v) \, dA.
\]

We choose the sign according to the specified orientation of \( S \).

Examples from Stewart:

\#16 \( \vec{F}(x, y, z) = x^2 \, y \, \hat{i} - 3 \, x \, y^2 \, \hat{j} + 4 \, y^2 \, \hat{k} \) \( S \) is the part of the paraboloid \( z = x^2 + y^2 - 9 \) below the rectangle \( 0 \leq x \leq 2, \ 0 \leq y \leq 1 \) and \( S \) is oriented downward.

\#18 \( \vec{F}(x, y, z) = -x \, \hat{i} - y \, \hat{j} + z^2 \, \hat{k} \) \( S \) is the part of the cone \( z = \sqrt{x^2 + y^2} \) between the planes \( z = 1 \) and \( z = 2 \). \( S \) is oriented upward.