Section 14.9 The Divergence Theorem

Recall the 2nd statement of Green’s Theorem:

\[ \oint_{C} F \cdot n \, ds = \iint_{D} \text{div} F \, dA \]

where \( C \) is a simple closed curve enclosing the region \( D \), and the components of the vector field, \( F \), have continuous partial derivatives on an open region containing \( C \) and \( D \).

The Divergence Theorem extends this to \( \mathbb{R}^3 \).

We assume: \( E \) is a simple solid region with boundary surface \( S \). \( S \) is a closed surface with outward orientation. The components of the vector field, \( F \), have continuous partial derivatives on an open region containing \( E \) and \( S \).

Then

\[ \iiint_{E} \text{div} F \, dV = \oiint_{S} F \cdot n \, dS \]

Examples:

1. \( \vec{F} (x, y, z) = ax \vec{i} + by \vec{j} + cz \vec{k} \) \( S \) is the surface of the sphere of radius \( A \).

Use the Divergence Theorem to find the flux of \( F \) across \( S \).

2. Find the flux of \( \vec{F} = x \vec{i} + y \vec{j} + 4 \vec{k} \) across the surface of the region bounded by \( x^2 + z^2 = 1, \ y = 0, \) and \( x + y = 3 \).

3. Find the flux of \( \vec{F} = x^3 \vec{i} + 2xz^2 \vec{j} + 3y^2 z \vec{k} \) across surface of the region bounded by \( z = 4 - x^2 - y^2, \) and \( z = 0 \).