151 Week in Review

4.3, 4.6, 4.8, 5.1 Solutions

1a) $\log_2(6x-4)=3$

$6x-4=2^3=8 \qquad 6x=12 \qquad \boxed{x=2}$

b) $\log_2[(x-4)(x-1)]=4$

$x^2-5x+4=2^4=16$

$x^2-5x-12=0$

$x=\dfrac{5\pm\sqrt{25+48}}{2}=\dfrac{5\pm\sqrt{73}}{2}$

$\dfrac{5-\sqrt{73}}{2}<0$ so $\boxed{x=\dfrac{5+\sqrt{73}}{2}}$

c) $\ln 4^{2x}=5 \qquad 2x\ln 4=5 \qquad \boxed{x=\dfrac{5}{2\ln 4}}$

2 a) $f'(x)=\dfrac{1}{1+x^2}$ b) $f'(x)=\left(\dfrac{1}{1+(\ln x)^2}\right)\left(\dfrac{1}{x}\right)$

c) $\ln f(x)=x^2\ln x$

$\dfrac{f'(x)}{f(x)}=2x\ln x+x^2\cdot\dfrac{1}{x}=2x\ln x+x$

$f'(x)=x^{x^2}\left[2x\ln x+x\right]$

2 d) $\ln f(x) = (\ln x)\ln x = (\ln x)^2$

$$\frac{f'(x)}{f(x)} = 2(\ln x)\frac{1}{x} = \frac{2\ln x}{x}$$

$$f'(x) = x^{(-1+\ln x)}(2\ln x)$$

3. $W(t) = W(0)\left(2^{\frac{t}{\text{doubling time}}}\right)$ $\quad$ $t = $ # of minutes past 28g ~~t~~ime

$$= 28\left(2^{t/30}\right)$$

Solve $28\left(2^{t/30}\right) = 100$

$$2^{t/30} = \frac{100}{28} = \frac{25}{7}$$

$$\frac{t}{30}\ln 2 = \ln\frac{25}{7}$$

$$t = \frac{30\ln\left(\frac{25}{7}\right)}{\ln 2} \approx 55.1 \text{ minutes}$$

4. $W(t) = W(0)\left(\frac{1}{2}\right)^{\frac{t}{\text{half-life}}}$ $\quad$ $t = $ # days past 500 mg date

$$= 500\left(\frac{1}{2}\right)^{t/140}$$

Solve $500\left(\frac{1}{2}\right)^{t/140} = 25$

$$\left(\frac{1}{2}\right)^{t/140} = \frac{25}{500} = \frac{1}{20}$$

$$\frac{t}{140}\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{20}\right)$$

$$t = \left(\frac{-\ln 20}{-\ln 2}\right)(140) \approx 605.07 \text{ days}$$

5a) Find $\frac{d}{dx}\left(\ln f(x)\right)$ first.

$$\ln f(x) = 2\ln(3x+1) + \frac{1}{3}\ln(x-5)$$
$$- \frac{1}{2}\ln(2x+3) - 5\ln(4x^2+2)$$

$$\frac{f'(x)}{f(x)} = 2\,\frac{3}{3x+1} + \frac{1}{3}\cdot\frac{1}{x-5} - \frac{1}{2}\cdot\frac{2}{2x+3}$$
$$-5\cdot\frac{8x}{4x^2+2}$$

$$= \frac{6}{3x+1} + \frac{1}{3(x-5)} - \frac{1}{2x+3} + \frac{4x}{2x^2+1}$$

$$f'(x) = \frac{(3x+1)^2\sqrt[3]{x-5}}{\sqrt{2x+3}\,(4x^2+2)^5}\left[\frac{6}{3x+1} + \frac{1}{3(x-5)} - \frac{1}{2x+3} + \frac{4x}{2x^2+1}\right]$$

5 b) $f'(x) = 2\tan x$

6 i) $f(x) = 2\ln|x| + x^3$ $\quad f(1) = 1$

$f'(x) = \frac{2}{x} + 3x^2$ $\quad f'(1) = 5$

tangent line: $y = 5(x-1) + 1$

$y = 5x - 4$

ii) Recall $\frac{d}{du}\left(\operatorname{arcsec} u\right) = \frac{1}{u\sqrt{u^2-1}}$

Using the chain rule:

$$f'(x) = \frac{1}{\sqrt{x}\sqrt{x-1}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{x-1}}$$

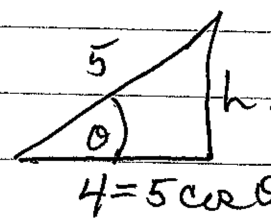
$f(2) = \operatorname{arcsec}\sqrt{2} = \pi/4$ $\quad f'(2) = \frac{1}{4}$ $\quad \boxed{y = \frac{1}{4}(x-2) + \frac{\pi}{4}}$

6 iii) $f\left(\frac{1}{\sqrt{3}}\right) = \arctan\frac{1}{\sqrt{3}} = \pi/6$

$f'(x) = \frac{1}{1+x^2}$ $\quad f'\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+1/3} = 3/4$

$y = \frac{3}{4}\left(x - \frac{1}{\sqrt{3}}\right) + \frac{\pi}{6}$

7. a) $\sin\left(\arccos\frac{4}{5}\right)$



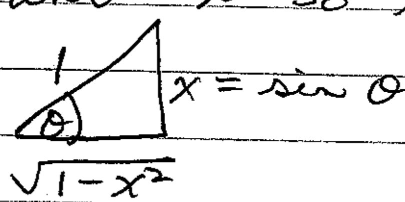

$h = \sqrt{25-16} = 3$ $\quad \sin\theta = \frac{h}{5} = \boxed{\frac{3}{5}}$.

Or: If $\cos\theta = \frac{4}{5}$, $\sin\theta = \sqrt{1-\frac{16}{25}} = 3/5$.

b) If $\sin\theta = x$, $\cos\theta = \boxed{\sqrt{1-x^2}}$

c) $\tan\theta = x$ so $\sec\theta = \sqrt{1+\tan^2\theta} = \boxed{\sqrt{1+x^2}}$

d)



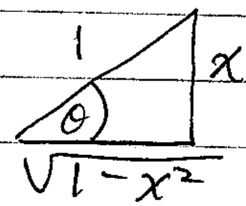

$\tan\theta = \boxed{\frac{x}{\sqrt{1-x^2}}}$

8\. a) $f'(x) = \frac{1}{2}(1-4x^2)^{-1/2}(-8x)\arcsin(2x)$

$$+ \sqrt{1-4x^2} \cdot \frac{2}{\sqrt{1-4x^2}}$$

$$= -\frac{4x \arcsin 2x}{\sqrt{1-4x^2}} + 2$$

b) Simplify first:




$\sec(\arcsin x)$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = \boxed{\frac{x}{(1-x^2)^{3/2}}}$$

c) $\frac{d}{du}(\arctan u) = \frac{1}{1+u^2}$

$$f'(x) = \frac{1}{1+e^{2x^2}} \cdot 2x e^{x^2} = \frac{2xe^{x^2}}{1+e^{2x^2}}$$

9. a) $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ Substitution gives the indeterminate form "$\frac{\infty}{\infty}$", L'Hospital's rule can be applied:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

b) Substitution the indet. form "$1^{\infty}$".

$$\ln\left[(1+4x)^{5/x}\right] = \frac{5}{x}\ln(1+4x)$$

$$= \frac{5\ln(1+4x)}{x}$$

Applying L'Hospital's Rule:

$$\lim_{x\to 0} \frac{5\ln(1+4x)}{x} = \lim_{x\to 0} \frac{\frac{20}{1+4x}}{1} = 20$$

$\therefore$

$$\lim_{x\to 0} (1+4x)^{5/x} = \boxed{e^{20}}$$

9c) Substitution gives "$\infty - \infty$" which is indeterminate. We must do algebra to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying L'Hospital's Rule.

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - (x-1)}{(x-1)\ln x}$$

Subst. gives "$\frac{0}{0}$"

$$\sim \frac{\ln x + 1 - 1}{\ln x + \frac{x-1}{x}} = \frac{\ln x}{\ln x + \frac{x-1}{x}} = \frac{\ln x}{\ln x + 1 - \frac{1}{x}}$$

"has the same limit as"

Substitution still gives "$\frac{0}{0}$"

$$\sim \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} \underset{\text{algebra}}{=} \frac{x}{x+1} \xrightarrow[x \to 1]{} \boxed{\frac{1}{2}}$$

d) Substitution gives "$\frac{0}{0}$".

Applying L'Hospital's Rule:

$$\frac{3^x - 4^x}{x} \sim \frac{3^x \ln 3 - 4^x \ln 4}{1} \xrightarrow[x \to 0]{} \ln 3 - \ln 4 = \boxed{\ln \frac{3}{4}}$$

9e) $\lim_{x\to\infty} \frac{1-x}{2x+3} = -\frac{1}{2}$. Since $\arccos u$ is continuous,

$$\lim_{x\to\infty} \arccos\left(\frac{1-x}{2x+3}\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

9f) Substitution gives the indeterminate form "$0^0$"

$$\ln(x^x) = x\ln x = \frac{\ln x}{\frac{1}{x}}$$

Now Substitution gives $\frac{-\infty}{\infty}$ so L'Hosp.'s Rule can be applied:

$$\lim_{x\to 0^+}(\ln x^x) = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \underset{\text{algebra}}{=} \lim_{x\to 0}(-x) = 0$$

So $\lim_{x\to 0^+} x^x = e^0 = 1$.

10 a) $f(x) = \ln(x^2+1)$

$$f'(x) = \frac{2x}{x^2+1}$$

sign $f'$: $-$ on $(-\infty, 0)$, $0$ at $x = 0$, $+$ on $(0, \infty)$

$f$: $\searrow$ $\nearrow$

local minimum at $(0, f(0)) = (0,0)$

Domain $f = (-\infty, \infty)$

$f$ is decreasing on $(-\infty, 0)$

$f$ is increasing on $(0, \infty)$ local min is 0 at $x=0$.

$$f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{1-2x^2}{(x^2+1)^2} = \frac{(1-\sqrt{2}x)(1+\sqrt{2}x)}{(x^2+1)^2}$$

sign $f''$: $-$ on $(-\infty, -\frac{1}{\sqrt{2}})$, $+$ on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $-$ on $(\frac{1}{\sqrt{2}}, \infty)$

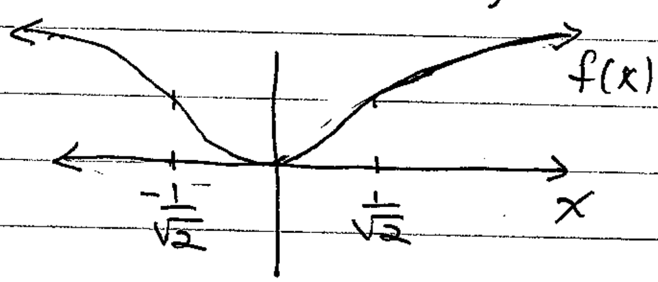
$f$: $\frown$ $\smile$ $\frown$

$f$ is concave down on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \infty)$

$f$ is concave up on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$f$ has inflection points

at $(-\frac{1}{\sqrt{2}}, \ln\frac{3}{2})$ and $(\frac{1}{\sqrt{2}}, \ln\frac{3}{2})$



10 b) $f(x) = e^{-x^2}$

Domain $f = (-\infty, \infty)$

$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$

$f'(x) = -2xe^{-x^2}$

sign $f'$   $+$   $0$   $-$   $x$

$f$   $\nearrow$   $\searrow$

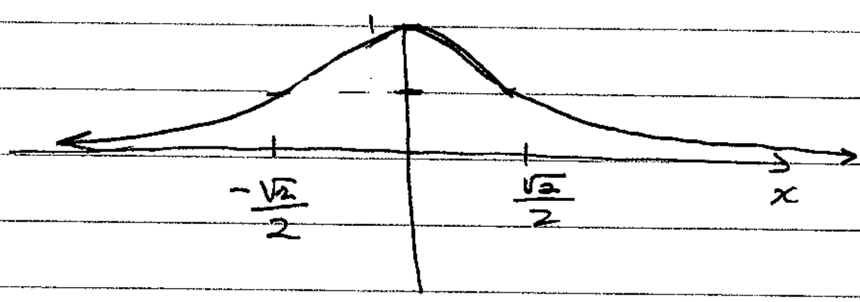
local max at $(0, 1)$

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$$
$$= (4x^2 - 2)e^{-x^2}$$

$4x^2 - 2 = (2x - \sqrt{2})(2x + \sqrt{2})$

sign $f''$   $+$   $-\frac{\sqrt{2}}{2}$   $-$   $\frac{\sqrt{2}}{2}$   $+$   $x$

$f$   $\cup$   $\cap$   $\cup$




A transformation of the normal bell curve.