

6.3 The Multiplication Principle

Perform a task with two steps. If there are m ways to do step 1 and n ways to do step 2, then there are mn ways to do both steps. Remember we are doing both steps and each way of doing step 1 can be paired with each way of doing step 2.

Ex. An urn contains 3 balls colored red, blue and green. A coin has two sides, H and T. We draw a ball from the urn and then toss the coin. How many results are possible? $3 \times 2 = 6$

We can list the possible results as ordered pairs with color first and coin side second. They are (red,H), (blue,H), (green,H),
(red,T), (blue,T), (green,T)

Ex. There are 4 science course, 3 electives, and 5 social science courses. A student must choose one from each category. We could list the possible choices as ordered triples but we don't want to-there would be $4 \times 3 \times 5 = 60$ of them.

Ex. A two sided coin is tossed n times. How many possible outcomes are there? 2^n
Write them down as ordered pairs for $n=2$ and as ordered triples for $n=3$.

Ex. How many 3-letter sequences can be made from the 26 letters of the alphabet if repetitions are allowed? 26^3

Ex. How many 3-letter sequences, repeats allowed, contain no a's? 25^3
How many 3-letter sequences, repeats allowed, contain at least one a? $26^3 - 25^3$

Ex. How many license plates have 3 letters followed by 4 digits if the 1st digit cannot be 0?
 $26^3 \times 9 \times 10^3$

The difference between adding and multiplying: Suppose we have two urns, A and B.
Urn A has 3 balls colored red, blue and green. Urn B has 5 balls numbered 1 through 5.
How many ways can we choose a ball from urn A and a ball from urn B? 15
How many ways can we choose one ball from either urn A or urn B? 8

6.4 Permutations and Combinations

Permutations: A permutation of a set A is an arrangement, or an ordered list, of the elements of A in which each element occurs exactly once. No repeats allowed. Order matters.

Ex. $A = \{a, b, c\}$ The permutations of A are abc, acb, bac, bca, cab, cba
Why are there six permutations of a 3-element set? We have 3 choices for the 1st letter. We cannot repeat this letter so there are only 2 choices for the 2nd letter. Then we have only one letter left for the 3rd letter. $3 \times 2 \times 1 = 6$

How many permutations of an n -element set are there? We have n choices for the 1st, $n-1$ choices for the 2nd, $n-2$ choices for the 3rd and so on until we have only one choice for the last.
There are $n \times (n-1) \times (n-2) \dots \times 2 \times 1$. This number is called n factorial and is written $n!$. It is on the calculator under MATH >> PRB. It is consistent to make $0!$ equal to 1.

Ex. How many different words can be made from the letters in {a, b, c, d, e} if no repeats are allowed? Here a word is any permutation of the letters. $5!=120$

Sometimes we do not arrange all the n elements of a set but only r of them where $r < n$.

Ex. How many 3 letter words can be made from the 26 letters of the alphabet if no repeats are allowed? There are 26 choices for the 1st letter. We cannot repeat this letter so there are 25 choices for the 2nd letter and we cannot repeat either of these letters so there are 24 choices for the 3rd. $26 \times 25 \times 24$ This is called $P(26,3)$ —the number of permutations of 26 things taken 3 a time.

$$P(n,r) = n(n-1)(n-2)\dots(n-r+1) = n!/(n-r)!$$

Ex. $P(7,3) = 7 \times 6 \times 5 = 7!/4!$ On the calculator enter 7 Math>>PRB select nPr enter 3

Ex. How many ways can a secretary, treasurer, vice president and president be chosen from a committee of 10 people?

Think of assigning the positions as arranging the four people in order. This is $P(10,4) = 5040$

Ex. How many distinct arrangements of the word banana are there?

Banana has six letters, 1-b, 3-a's, and 2-n's. If we could distinguish all the letters there would be $6!$ Permutations. But since we can't distinguish the $3!$ permutations of the a's and the $2! = 2$ permutations of the n's, $6!$ counts each word $3! \times 2!$ times. There are $6!/(3!2!)$ distinguishable permutations.

Ex. How many arrangements of Mississippi are there? $11!/(4!4!2!)$

Combinations: A combination is a subset. How many ways can we choose r element subset from an n -element set, $r \leq n$. Order does not matter. No repeats.

Ex. How many 3-element subsets of the 26 letter alphabet are there?

Ex. There are 10 people on a committee. How many ways can we choose 4 of them to serve on a special project? Compare these examples to the permutations done previously.

How do we answer these questions? Let's start with the 3-element subset of the 26 letter alphabet. We know there are $P(26,3) = 26!/23!$ different ways to arrange three out of 26 letters.

How many times does this count the combination {a,b,c}? {a,b,c} has been counted $3!$ times, once for each different ordering. So also {r,b,l} has been counted $3!$ times. Every 3-element subset has been

counted $3!$ times, once for each arrangement of its letters. The number we want is $P(26,3)/3! = \frac{26!}{23!3!} = 2600$

$$C(n,r) = \frac{n!}{(n-r)!r!} \text{ the number of combinations of } n \text{ things taken } r \text{ at a time} = \text{the number of}$$

r -element subsets of an n -element set. This is under MATH>>PRB select nCr on the calculator

Ex. The number of ways to choose a subcommittee of 4 from a committee of 10 is $C(10,4) = 210$ Here are some examples of permutations and combinations:

Permutation

Form a line of 5 people from a room of 50. $P(50,5)$

Paint 4 different objects with any of 10 different colors, no repeats. $P(10,4)$

Choose 3 out of 15 building sites for a 1,500 sq ft home,

2,000 sq ft home and a 3,000 sq ft home. Decide which home goes on which lot. $P(15,3)$

Combination

Choose a committee of 5 people out of 50 $C(50,5)$

Choose 4 of 10 colors. $C(10,4)$

Choose 3 out of 15 lots for 3 new homes. $C(15,3)$

Examples:

1. A landscaper has 15 species of flowers in 3 colors. There are 6 blue species, 5 white species, and 4 gold species. How many ways can they be arranged if like colors must be together? A) Choose the color arrangement. $3!$ ways
B) Arrange the species in each color, $6!, 5!, 4!$
C) Apply the multiplication principle $3!6!5!4!=12441600$
2. A crate of oranges has 47 good oranges and 3 bad ones. How many ways can you select 7 oranges so that 5 are good and 2 are bad? Select 5 good ones in any of $C(47,5)$ ways and select 2 bad ones in any of $C(3,2)$ ways. Use the multiplication principle. $C(47,5)C(3,2)$
3. How many ways can you arrange 10 boys and 6 girls so that no two girls are next to each other? Arrange the 10 boys in any of $10!$ ways. The girls have to stand in the 11 spaces left by the boys. How many ways can we assign 6 girls to any of 11 spaces? $P(11,6)$. Now apply the multiplication principle. $10!P(11,6)$.