

Probability II

Conditional Probability

You already know probabilities change when more information is known. For example the probability of getting type I diabetes for the general population is .06. The probability of getting type I diabetes if you have an identical twin with diabetes is .5. The sample space in the 2nd case is only those who have an identical twin with diabetes.

Another example is auto accident rates. The probability of an auto accident is higher for people in the 18-25 year age group. If we reduce the sample space to the 18-25 year age group we get a different probability.

Suppose someone rolls a fair die and we have to guess what number showed on top. With no information we have a 1/6 probability of getting it right. But if they tell us the number was even, we know the number was either 2, 4, or 6 and each of these has a 1/3 probability, whereas 1, 3, and 5 now have 0 probability. We have a 1/3 probability of getting it right. The conditional probability of the number being a 2 given that the number was even is 1/3.

The conditional probability of event B given that event A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \quad \text{Notice that this means } P(A \cap B) = P(A)P(B|A)$$

$$\text{In a uniform probability space, } P(B|A) = \frac{n(A \cap B)}{n(A)}$$

Ex. Someone rolls two dice and tells us the sum of the top numbers is 8. What is the probability the numbers are 3 and 5?

The given event is $A = \{(2,6) (3,5) (4,4) (5,3) (6,2)\}$.

$B = \{(3,5) (5,3)\}$

$P(B|A) = 2/5$

Ex. Someone rolls two dice. What is the probability the sum was 4 given at least one of the numbers was a 3?

The given event is $A = \{(1,3) (2,3) (3,3) (4,3) (5,3) (6,3) (3,1) (3,2) (3,4) (3,5) (3,6)\}$

$A \cap B = \{(1,3), (3,1)\}$

$P(B|A) = 2/11$

Ex. The table shows the breakdown as male-female, smoker-nonsmoker for a certain group of 100 people. We will use the relative frequency distribution.

	smoker	nonsmoker	total
male	15	25	40
female	10	50	60
total	25	75	100

A person is randomly chosen from the group. Given the person is a male, what is the probability he is a smoker? More shortly, what is the probability a male is a smoker?
 $P(\text{smoker}|\text{male}) = 15/40$ because there are 40 males and 15 of them are smokers.

What is the probability a smoker is a male? Now the given is that he is a smoker.

There are 25 smokers and 15 are male so $P(\text{male}|\text{smoker}) = 15/25$

What is the probability a nonsmoker is a female? $P(\text{female}|\text{nonsmoker})=50/75$

What is the probability the person is a male smoker? No information is given about the person. This is not a conditional probability. $P(\text{male and smoker})=15/100$

Ex. Suppose there is a medication to help prevent a certain disease. In a sample of 100 people, 65 took the medication and 5 of these got the disease. The other 35 took no medication and 25 of these got the disease. Using the relative frequency distribution, find the probability a person who takes the medication will not get the disease.

The given condition is that the person took the medication. There are 65 of these.

How many of these did not get the disease? $65-5=60$

$P(\text{no disease}|\text{medication})=60/65$

The table can help with other conditional probabilities.

	Medication	No medication	total
Disease	5	25	30
No Disease	60	10	70
	65	35	100

What is the probability someone who did not get the disease took the medication?

The given condition is the person did not get the disease. There are 70 of these.

60 of the 70 took the medication so $P(\text{medication}|\text{no disease})=60/70$

What is the probability someone took the medication and did not get the disease? $60/100$

No information about the person was given.

Make two tree diagrams for this information.

Ex. In this case the conditional probabilities are easy to see and the unconditional probabilities can be found from them.

We have two urns containing colored balls, urn I and urn II. Urn I has 3 red and 5 blue balls. Urn II has 2 red and 3 blue balls. We choose an urn and then choose a ball. The urns are equally likely to be chosen. What is the probability of getting a red ball.

If we know the ball is to come from urn I then the probability of red is $3/8$. $P(\text{red}|\text{I})=3/8$

If we know the ball is to come from urn II then the probability of red is $2/5$. $P(\text{red}|\text{II})=2/5$

The ball has to come from urn I or urn II and it can't come from both so

$$P(\text{red})=P(\text{red and I})+P(\text{red and II})=P(\text{red|I})P(\text{I})+P(\text{red|II})P(\text{II})=\frac{3}{8} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{2}=31/80$$

Make a tree diagram for this experiment. Use the tree to find P(red) again.

Ex. A medical test tests for the presence of a certain disease. Among those who have the disease, the test is negative (it fails to detect the disease) 5% of the time. Among those who do not have the disease, the test is positive 6% of the time. Four percent of the population has the disease. Later we will answer the question “What is the probability someone who gets a positive result actually has the disease?”. For now we will find the probability that the test is positive.

Make a tree diagram: Let + be the event that the test is positive and let D be the event that the person has the disease.

$$P(+)=P(+\text{and } D) + P(+ \text{ and } D^c)=P(+|D)P(D) + P(+|D^c)P(D^c)=.95 \times .04 + .06 \times .96=.0956$$

Independent Events

Events A and B are independent means $P(A \cap B) = P(A)P(B)$.

If P(A) is not zero then this means $P(B|A)=P(B)$ and if P(B) is not zero, $P(A|B)=P(A)$ so knowing one has occurred does not change the probability of the other.

Ex. Successive tosses of a coin are independent. If we toss a coin two times the probability of getting heads on the second toss has nothing to do with what we got on the

$$\text{first toss. } P(\text{H on 1}^{\text{st}} \text{ toss and H on 2}^{\text{nd}} \text{ toss})=\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Ex. In 1000 people, 500 took supplement S to try prevent condition D. The other 500 took nothing. The results were that 760 of the 1000 got condition D. 380 of those taking the supplement got condition D. Are taking the supplement and getting condition D

independent? $P(D)=.76$ $P(D|S)=380/500=.76$ so they are independent according to this study.

Ex. A group of 100 people took a test. 50 of them took a pre-test and 50 did not. The results of how many passed the test and how many failed were

	pre-test	no pre-test	total
Passed	48	20	68
Failed	2	30	32
Total	50	50	100

Are taking the pre-test and passing the test independent?

$P(P \cap S) = .48$ $P(P) = .68$ $P(S) = .5$ $.48$ does not equal $.68 \times .5$ so they are not independent.

Bayes' Theorem: When you have $P(B|A)$ and $P(B|A^c)$ and $P(A)$ and you want $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$
 Follow this in the tree diagram:

Example: Refer to the example earlier with the medical test for a certain disease.

$P(D) = .04$, $P(+|D) = .95$, and $P(+|D^c) = .06$

Suppose you get a positive test. What is the probability you actually have the disease?

That is, what is $P(D|+)$?

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)} = \frac{.95 \times .04}{.95 \times .04 + .06 \times .96} = .39748 \dots$$

Make the tree diagram:

Example: A company manufactures widgets in three factories I, II, and III. A portion of the widgets produced by each factory are defective. The production rates and defective rates are given in the table.

Factory	% of total	% of defective
I	35	3
II	40	4
III	25	5

So factory I makes 35 % of the widgets and 3% of those made by factory I are defective. Given a defective widget, what is the probability it came from factory I? Make a tree diagram or use the table. Let D be the event that the widget is defective.

$$P(I|D) = \frac{.35 \times .03}{.35 \times .03 + .40 \times .04 + .25 \times .05} = .26923\dots$$

Review:

Example: Urn I contains 3 red and 6 green balls. Urn II contains 2 red and 4 green balls. An urn is chosen and a ball is drawn from that urn. Assume the probability of choosing urn I is .5.

Find $P(\text{red}|I)$, $P(\text{red}|II)$, $P(\text{red})$

Find the probability urn I and a red ball were chosen, $P(I \text{ and red})$

$$\text{Find } P(I|\text{red}) = \frac{P(I \text{ and red})}{P(\text{red})}$$

Make a tree diagram.

Using the tree answer the last question again:

Given that a red ball was chosen, what is the probability it came from urn I? $P(I|\text{red})$