Solutions WIR 151 2.5–2.7

2. Find $\lim_{x\to 2} f(x)$, if it exists.

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{\cancel{x-2}} = 2-1=1$$

But $f(2)=4$ so f is not continuous at $x=2$.

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3. We must have $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}(x^2-c) = 25-c$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}(4x+2c) = 20+2c$$

$25-c = 20+2c$ if $5=3c$, $\boxed{c = 5/3}$

4. If $f(2)$ and $f(3)$ have opposite sign, then the intermediate value theorem says f has a root between 2 and 3, if f is continuous.

$f(x) = x^3-8x-1$ $f(2)=-9$ $f(3)=2$

and all polynomials are continuous.

5. $\dfrac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} = \dfrac{1 - 2/x + 3/x^2 - 4/x^3}{4 - \frac{3}{x} + 2/x^2 - 1/x^3}$

So the limit as $x \to \infty$ is $\boxed{1/4.}$

6. $\dfrac{x^3 - 2x^2 + 3x - 4}{4x^4 - 3x^2 + 2x - 1} = \dfrac{1 - 2/x + 3/x^2 - 4/x^3}{4x - 3/x + 2/x^2 - 1/x^3}$

So the limit as $x \to \infty$ is 0.

5 & 6 can be done as follows:

5) $\dfrac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} \sim \dfrac{x^3}{4x^3} \xrightarrow[x \to \infty]{} 1/4$

6) $\dfrac{x^3 - 2x^2 + 3x - 4}{4x^4 - 3x^2 + 2x - 1} \sim \dfrac{x^3}{4x^4} = \dfrac{1}{4x} \xrightarrow[x \to \infty]{} 0$

For rational functions, we only need to look at the ratio of the leading terms as $x \to \infty$.

7. " $\infty - \infty$ " is indeterminate. Do algebra.

Conjugating we have:

$$\frac{x^2-3x+2 \; - \; x^2}{\sqrt{x^2-3x+2} \; + \; x} = \frac{-3x+2}{\sqrt{x^2-3x+2}+x}$$

$$* \quad = \frac{-3+2/x}{\sqrt{1-3/x+2/x^2}+1} \xrightarrow[x\to\infty]{} -3$$

*(Divide numerator and denominator by x.

$$\frac{1}{x}\sqrt{x^2-3x+2} = \sqrt{\frac{x^2-3x+2}{x^2}} = \sqrt{1-3/x+2/x^2}\,)$$

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8. $$\lim_{x\to\infty}\frac{\sqrt{x^2+4x}}{2x-7} = \lim_{x\to\infty}\frac{\sqrt{1+4/x}}{2-7/x} = \frac{1}{2}$$

$y=\frac{1}{2}$ is the horizontal asymptote

as $x \to +\infty$.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+4x}}{2x-7} = \frac{-\sqrt{1+4/x}}{2-7/x} = -1/2$$

since $\frac{1}{x} = -\sqrt{1/x^2}$

$y=-1/2$ is the horizontal asymptote

as $x \to -\infty$

9. I $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is the limit definition of the slope of the tangent line.

Conjugating we have:

$$\frac{\sqrt{2(x+h)+3} - \sqrt{2x+3}}{h} = \frac{2(x+h)+3-(2x+3)}{h\left(\sqrt{2(x+h)+3}+\sqrt{2x+3}\right)}$$

$$= \frac{2h}{h\left(\sqrt{2(x+h)+3} + \sqrt{2x+h}\right)}$$

$$= \frac{2}{\sqrt{2(x+h)+3}+\sqrt{2x+3}} \xrightarrow[h \to 0]{} \frac{2}{2\sqrt{2x+3}}$$

$$= \frac{1}{\sqrt{2x+3}}$$

Plug in $x=3$ to get the slope at $x=3$.

$$\frac{1}{\sqrt{6+3}} = \frac{1}{3}$$

The tangent line is:

$$y = \tfrac{1}{3}(x-3)+3 = \tfrac{1}{3}x+2$$

II $$\frac{\sqrt{2(3+h)+3} - 3}{h} = \frac{\sqrt{9+2h} - 3}{h}$$

$$= \frac{9+2h-9}{h\left(\sqrt{9+2h}+3\right)} \xrightarrow[h \to 0]{} \frac{2}{2(3)} = 1/3$$

10. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{4}{x+h} - \frac{4}{x}}{h} = \dfrac{\frac{4x-4(x+h)}{x(x+h)}}{h}$

$= \dfrac{4x-4(x+h)}{h\cdot x(x+h)} = \dfrac{-4h}{hx(x+h)} = \dfrac{-4}{x(x+h)}$

$\lim\limits_{h\to 0} \dfrac{-4}{x(x+h)} = \dfrac{-4}{x^2}$ if $x \neq 0$.

Plug in $x=2$. $m = \dfrac{-4}{2^2} = -1$

The tangent line at $x=2$ is

$y = -(x-2) + 2 = -x+4$

$f(2)=2$

11 a) $$\frac{\vec{r}(3)-\vec{r}(1)}{3-1} = \frac{9\vec{i}+12\vec{j}-(5\vec{i}+2\vec{j})}{3-1}$$

$$= 2\vec{i}+5\vec{j}$$

b) $$\frac{\vec{r}(3+h)-\vec{r}(3)}{h} = \frac{[2(3+h)+3]\vec{i}+[(3+h)^2+(3+h)]\vec{j}-[9\vec{i}+12\vec{j}]}{h}$$

$$= \frac{(9+2h)\vec{i}+(9+6h+h^2+3+h)\vec{j}-[9\vec{i}+12\vec{j}]}{h}$$

$$= \frac{2h\vec{i}+(7h+h^2)\vec{j}}{h} = 2\vec{i}+(7+h)\vec{j}$$

$$\xrightarrow[h\to 0]{} 2\vec{i}+7\vec{j}$$