ON TANGENTIAL VARIETIES OF RATIONAL HOMOGENEOUS VARIETIES

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Abstract. We determine which tangential varieties of homogeneously embedded rational homogeneous varieties are spherical. We determine the homogeneous coordinate rings and rings of covariants of the tangential varieties of homogenously embedded compact Hermitian symmetric spaces (CHSS). We give bounds on the degrees of generators of the ideals of tangential varieties of CHSS and obtain more explicit infomation about the ideals in certain cases.

1. Introduction

Let K be an algebraically closed field of characteristic zero, let G be a semisimple algebraic group over K, and let P be a parabolic subgroup. Consider the homogeneous space $X = G/P$ embedded equivariantly as the orbit of a highest weight line in a projective space $\mathbb{P}V$, where V is an irreducible G-module.

We investigate properties of the *tangential variety* $\tau(X) \subset \mathbb{P}V$ which is, by definition, the union of the points on the embedded tangent lines (i.e. $\mathbb{P}^1$'s) to X.

In section 2, we determine when $\tau(X)$ is spherical (Theorem 1.1). We show $\tau(X)$ is G-spherical iff X admits the structure of a compact Hermitian symmetric space (CHSS) (possibly for a larger group $G' \supseteq G$) except for the case that X contains G_2/P_1 as a factor. The results of this section are proved using properties of the projective second fundamental form and combinatorics of root systems, with the exception of the case where G is of type A_n, where we use an additional argument.

For the remainder of the paper we restrict to the case where V is a *generalized cominuscule* G-module. One can define a generalized cominuscule module V by the property that for $G/P = X \subset \mathbb{P}V$, the tangent bundle TX is an irreducible homogeneous vector bundle. With this definition, a *cominuscule module* is a generalized cominuscule module where moreover G is simple and V is a fundamental representation. We refer to $X \subset \mathbb{P}V$ as a *(generalized) cominuscule variety*. A generalized cominuscule variety admits the structure of a G-compact Hermitian symmetric space, and when we refer to the *rank* of a generalized cominuscule variety, we mean the rank of the corresponding CHSS. Note that the rank can be defined purely in terms of the projective structre as the length of the osculating sequence of X, see [12].

In sections 3,4 and 5, assuming X is generalized cominuscule, we study the coordinate rings of its tangential variety $\tau(X)$. We use the fact that in the case when $\tau(X)$ is non-degenerate, the cone over it has a natural desingularization which is the total space of homogeneous vector bundle. We use methods from [20], which reduce the calculation of syzygies to the calculation of sheaf cohomology groups of certain vector bundles. We calculate some of this cohomology to determine the the coordinate ring of $\tau(X)$ as a G-module (Theorems 3.11, 5.2), which turns out to be uniform over cominuscule varieties of the same rank.

In section 6 we apply results from §3,§4 and §5, combined with deformation results of Grosshans [8] to deduce upper bounds on the degrees of generators of the defining ideals of $\tau(X)$ when X is generalized cominuscule.

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Finally in section 7 we draw consequences from the previous sections to give results on the defining ideals, describing certain cases explicitly and bounding the degrees of the generators of the ideals in all cases.

We describe the content in more detail below.

1.1. **Notation.** For a variety $Z \subset \mathbb{P}V$, we let $\hat{Z} \subset V$ denote the corresponding cone. For $z \in Z$ a smooth point, we let $\hat{T}_z Z \subset V$ denote the affine tangent space, $T_z Z$ the Zariski tangent space, and $\tilde{T}_z Z = \mathbb{P}(\hat{T}_z Z) \subset \mathbb{P}V$ the embedded tangent projective space. The two are related by $T_z Z = (\hat{T}_z Z/\hat{z}) \otimes \hat{z}^*$. Let K, G, P, X be as in the first paragraph. We use German letters to denote Lie algebras associated to algebraic groups. We use the ordering of roots as in [3]. The fundamental weights and the simple roots of $\mathfrak{g}$ are respectively denoted ω_i and α_i. P_k denotes the maximal parabolic of G obtained by deleting the root spaces corresponding to negative roots having a nonzero coefficient on the simple root α_k. More generally, for $J = (j_1, ..., j_s)$, P_J denotes the parabolic obtained by deleting the negative root spaces having a nonzero coefficient on any of the simple roots $\alpha_{j_1}, ..., \alpha_{j_s}$. $\Lambda_{\mathfrak{g}}, \Lambda_G$ respectively denote the weight lattices of $\mathfrak{g}$, G, and $\Lambda^+_{\mathfrak{g}} \subset \Lambda_{\mathfrak{g}}$, $\Lambda^+_G \subset \Lambda_G$ the dominant weights. We let $L \subset P$ be a (reductive) Levi factor and $\mathfrak{f} = [\mathfrak{l}, \mathfrak{l}]$ a semi-simple Levi factor. We write $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent.

When dealing with $\mathfrak{a}_n$-modules we sometimes use partions to index highest weights, with the dictionary $\pi = (p_1, ..., p_{n+1})$ corresponds to the weight $(p_1 - p_2)\omega_1 + (p_2 - p_3)\omega_2 + \cdots + (p_n - p_{n+1})\omega_n$.

1.2. **Sphericality.** Recall that a normal projective G-variety Z is *G-spherical* if some (and hence any) Borel subgroup B of G has a dense orbit in Z. (We emphasize the group G in our terminology because certain varieties we study will be G-varieties for several different groups.) Equivalently, Z is spherical if for all degrees d, $K[Z]_d$, the component of the coordinate ring of Z in degree d, is a multiplicity free G-module, see [1]. Note that this property for $\tau(X)$ *a priori* depends both on G and the embedding of X.

Theorem 1.1. *Let $X = G/P \subset \mathbb{P}V$ be a homogeneously embedded rational homogeneous variety. Then $\tau(X)$ is G-spherical iff X admits the structure of a CHSS, and no factor of X is G_2/P_1.*

Note that the varieties C_n/P_1, B_n/P_n, which are not cominuscule, have spherical tangential varieties.

1.3. **Ideals and singularities.** For any smooth variety $X \subset \mathbb{P}V$, if the tangential variety is *strongly nondegenerate* in the sense that a general point of $\tau(X)$ lies on a unique tangent line, then $\tau(X)$ admits a natural desingularization by the projective bundle $\tilde{T}X$, whose fiber over $x \in X$ is the embedded tangent projective space $\tilde{T}_x X \subset \mathbb{P}V$. The associated vector bundle $\hat{T}X$ is a subbundle of the trivial bundle $V \otimes \mathcal{O}_X$. It is a desingularization of the affine cone $\hat{\tau}(X) \subset V$. In the case X is homogenous and homogeneously embedded, this desingularization is an example of what Kempf [11] called "the collapsing of a homogeneous vector bundle". We apply the methods of [20] to study the cone $\hat{\tau}(X)$ via this desingularization in the cases of generalized cominuscule varieties. The essential point is that the minimal free resolution of the ideal of $\hat{T}X \subset V \otimes \mathcal{O}_X$ is given by a Koszul complex, and we can "push down" this information to get information about the ideal of $\tau(X)$.

Proposition 1.2. *Let $X = G/P \subset \mathbb{P}V$ be a homogeneously embedded rational homogeneous variety. Then $\tau(X)$ is strongly nondegenerate except in the case when X is generalized cominuscule, has rank two and the embedding is the minimal homogeneous embedding, or $X = \mathbb{P}^n$ and the embedding is minimal or quadratic Veronese.*

This proposition is essentially "known to the experts" because the first candidates for non-strongly degenerate $\tau(X)$, namely the rank 3 cominuscule varieties appearing in the third row of the Freudenthal magic chart, are known to be strongly nondegenerate (in fact they are "one apparent double point"). In any case it is easy to verify on a case by case basis.

In other words, for cominuscule varieties $X \subset \mathbb{P}V$, $\tau(X)$ is strongly nondegenerate except when it coincides with the secant variety of X, $\sigma(X)$, which is the Zariski closure of all points on all secant lines to X. The cominuscule rank two case is well understood, see §7.3.

Recall that a variety Y over a field of characteristic zero has *rational singularities* if it is normal and it admits a desingularization $\pi : Z \to Y$ such that $R^i\pi_*\mathcal{O}_Z = 0$ for $i > 0$.

Theorem 1.3. *Let $X \subset \mathbb{P}V$ be a rank $r \geq 3$ cominuscule variety. Then*

1. *$\tau(X)$ is normal, with rational singularities.*
2. *The coordinate ring $K[\tau(X)]$ has a uniform decomposition into irreducible modules given by theorem* (3.11)*.1 in the irreducible case and theorem* (5.2)*.1 in the reducible case.*
3. *The ring of covariants of the coordinate ring $K[\tau(X)]$ has generators described by theorem* (3.11)*.2 in the irreducible case and theorem* (5.2)*.2 in the reducible case.*
4. *If X is irreducible, the ideal of $\tau(X)$ is generated in degrees at most $4r - 4$ and if it is reducible with its highest rank factor having rank r_0, then the ideal of $\tau(X)$ is generated in degrees at most $\max(6, 4r_0)$.*

Our uniform degree bound follows from adapting a deformation argument applicable to spherical varieties. This idea dates back to unpublished work of Luna and to Brion's thesis [1]. We use the version of Grosshans [8]. This is explained in §6.

In §4 we describe the generators of rings of covariants for the tangential varieties of multiple embeddings of X.

We give more precise information about the ideals and singularities in some cases in §7.

We summarize from [20] (5.1.1-3,5.4.1) the results we will use:

Theorem 1.4. [20] *Let $Y \subset \mathbb{P}V$ be a variety and suppose there is a projective variety B and a vector bundle $E \to B$ that is a subbundle of a trivial bundle $\underline{V} \to B$ with $\underline{V}_z \simeq V$ for $z \in B$ such that $\mathbb{P}E \to Y$ is a desingularization of Y. Write $\eta = E^*$ and $\xi = (\underline{V}/E)^*$.*

If the sheaf cohomology groups $H^i(B, S^d\eta)$ are all zero for $i > 0$ and if the linear maps $H^0(B, S^d\eta) \otimes V^* \to H^0(B, S^{d+1}\eta)$ are surjective for all $d \geq 0$, then

1. *$\hat{Y}$ is normal, with rational singularities*
2. *The coordinate ring $K[\hat{Y}]$ satisfies to $K[\hat{Y}]_d \simeq H^0(B, S^d\eta)$.*
3. *The vector space of minimal generators of the ideal of Y in degree d is isomorphic to $H^d(B, \Lambda^{d+1}\xi)$.*
4. *If moreover Y is a G-variety and the desingularization is G-equivariant, then the identifications above are as G-modules.*

In our situation $\xi := (V \otimes \mathcal{O}_X/\hat{T}X)^*$ and $\eta := (\hat{T}X)^*$. The bundles ξ and η are homogeneous and indecomposable but not irreducible, so we first calculate the cohomology of the corresponding graded bundles using the Bott-Borel-Weil theorem and then pass to the cohomology we are interested in using methods from [19].

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2. DETERMINATION WHEN $\tau(X)$ IS SPHERICAL

In this section we reduce the calculation of whether or not $\tau(X)$ is spherical to a calculation if a linear representation is spherical, i.e., *visible* in the language of [10]. Then we use the list in [10] to determine the cases where P is maximal, and we determine other cases when P is not maximal by hand.

For any smooth variety $X \subset \mathbb{P}V$, let $x \in X$ be a general point and let $w \in T_xX$ be a generic vector. Then $\dim \tau(X) = \dim X + \operatorname{rank} II_w$, where $II \in S^2T^*_xX \otimes N_xX$ is the projective second fundamental form of X at x and $II_w : T_xX \to N_xX$ is the map $u \mapsto II(u, w)$, see [6] or [9], Proposition 3.13.3.

In our situation, let $x = [v]$ correspond to a highest weight vector. Then $T_xX \simeq \mathfrak{g}.v \bmod v$ (up to a twist by a line bundle which we ignore throughout this section). Let $U_1, U_2 \in \mathfrak{g}$, then $II(U_1.v, U_2.v) = U_1.(U_2.v) \bmod \mathfrak{g}.v$, see [12], Proposition 2.3.

Say the parabolic $\mathfrak{p}$ is obtained by deleting the root spaces of negative roots having nonzero coefficient on the simple roots $\beta_1, ..., \beta_p$. These simple roots induce a $\mathbb{Z}^p$ grading on $\mathfrak{g}$ which induces multi-filtrations on T_xX and $N_xX \simeq V/\hat{T}_xX$.

Let $L \subset P$ be a Levi factor, i.e., a maximal reductive subgroup of P, and write $T = T_xX = \oplus_I T_I$ for the decomposition of T as an L-module. For $I = (i_1, ..., i_p)$, T_I is the sum of root spaces for roots γ such that when γ is expressed as sum of simple roots that i_j is the coefficient of β_j in the expression.

A G-variety $Z \subset \mathbb{P}V$ is G-spherical iff there exists a Borel $B \subset G$ such that there is an open B orbit in Z, or equivalently, letting $z \in Z$ be a general point, and $\overline{z} \in V$ a corresponding vector in the line $z \in \mathbb{P}V$, Z is G-spherical iff $\mathfrak{b}.\overline{z} = \hat{T}_zZ \subset V$, where $\hat{T}_zZ$ is the affine tangent space.

Note that $\hat{\tau}(G_2/P_1) = V_{\omega_1}$ and V_{ω_1} is not a visible (spherical) G_2-module by [10].

We choose the Borel B containing the negative roots. We compare the spaces $\hat{T}_{v+U.v}\tau(X)$ and $\mathfrak{b}.(v + U.v)$ inside V. An arbitrary element of $\hat{T}_{v+U.v}\tau(X)$ is of the form $cv + U_1.v + U_2.U.v$ for $c \in \mathbb{C}$, $U_1, U_2 \in \mathfrak{g}$ and without loss of generality we may take $U_1, U_2 \in \mathfrak{g}_-$.

Let $\mathfrak{b}_0 \subset \mathfrak{g}_0$ denote the component of $\mathfrak{b}$ in $\mathfrak{g}_0$ so $\mathfrak{b} = \mathfrak{b}_0 + \sum_I \mathfrak{g}_{-I}$. For $b \in \mathfrak{b}$, write $b = b_0 + b_I$ with $b_0 \in \mathfrak{b}_0$.

Similarly, write $U = \sum U_I$ with $U_I \in \mathfrak{g}_{-I}$, so

$$U.v = \sum U_I.v$$

with $U_I.v \in T_I$. Now consider $U_2.U.v$.

Assume that $\tau(X)$ is of its expected dimension so $II_{U.v}$ is injective. (The cases where $\tau(X)$ is not of the expected dimension are always spherical (excepting G_2/P_1), which will follow by re-embedding the variety in such a way that $\tau(X)$ is nondegenerate by the argument below.) Then each $(U_2)_I$ must map injectively on $U.v$ and the vectors $(U_2)_I U_J.v$ must be independent as elements of N_xX.

Thus each b_J is used to fill $II_{U.v}(T) \subset N_xX$ exactly, and the only question that remains is if the vectors

$$b_0.U.v$$

fill $\hat{T}_xX \bmod \hat{x}$. Since $b_0.v \subset \hat{v}$, we are reduced to considering $[b_0, U]$. We have proved:

Lemma 2.1. *Let $X = G/P$ be homogeneous. Assuming $\tau(X)$ is nondegenerate, it is spherical iff for some $U \in \mathfrak{g}_-$, that $[\mathfrak{b}_0, U] = \mathfrak{g}_-$.*

Lemma 2.1 reduces the problem to a linear problem: determining which T_xX, considered as L-modules, are L-spherical, where $L \subset P$ is a Levi factor. The spherical irreducible L-modules were already determined by Kac [10], theorem 3. Examining his list we immediately conclude all cominuscule X have $\tau(X)$ spherical when $\tau(X)$ is nondegenerate. But for the same $X = G/P$ in a smaller embedding, this implies that $\tau(X)$ is also spherical in the smaller embedding. Since the only examples of homogeneous rational varieties with degenerate secant varieties occur among the cominuscule varieties and G_2/P_1, for all remaining cases we only need determine if T_xX is L-spherical. Recall the notation P_I from §1.1.

Proposition 2.2. *If $\tau(G/P_I)$ is not spherical, then $\tau(G/P_J)$ is not spherical for any set J of simple roots containing the simple roots corresponding to I.*

Proof. This is clear because $\mathfrak{b}_0^I \supset \mathfrak{b}_0^J$ but $\mathfrak{g}_-^I \subset \mathfrak{g}_-^J$. □

Kac's list already eliminates all non-cominuscule X except for: A_n/P (any P), B_n/P_n, $B_n/P_{1,n}$, C_n/P (some P, see below), $D_n/P_{1,n}$, F_4/P_4, G_2/P_1.

Among these F_4/P_4 is immediately eliminated by dimension considerations as $\mathfrak{l}$ is $\mathfrak{b}_3 \oplus \mathbb{C}$ and the L-module is $V_{\omega_1} \oplus V_{\omega_3}$, see [12] .

Proposition 2.3. *$\tau(B_n/P_n)$ is spherical.*

Proof. $T_x(B_n/P_n) \simeq \mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$ and we must examine the action of $\mathfrak{b}_0 \supset \mathfrak{b}(\mathfrak{sl}_n)$ on it. Consider the vector $v = e_1 \oplus e_1 \wedge e_2$, the sum of two highest weight vectors. It is clear $\mathfrak{b}(\mathfrak{sl}_n).v$ is $T_x(B_n/P_n)$. □

Proposition 2.4. *The tangential varieties of $D_n/P_{1,n}$ and $B_n/P_{1,n}$ are not spherical.*

Proof. For the $D_n/P_{1,n}$ case T_xX splits into three L-modules. Letting $V = V_{\omega_1} = \mathbb{C}^{n-2}$, they are $V \oplus V_{\omega_{n-2}} \oplus V_{\omega_1+\omega_{n-2}}$. Let $(v, U, Z) \in T_xX$. In order to have $\mathfrak{b}_0$ cover T_1, we must have $v = e_1$. Similarly to cover $T_{1,1}$ we must have Z contain a summand of the form $e_1 \otimes (e_1 \wedge F)$. But now we see it is impossible to have a vector of the form $(e_2, U, e_1 \wedge F)$ in $\mathfrak{b}_0.(v, U, Z)$. The B_n case is similar. □

Proposition 2.5. *The only homogeneous C_n-varieties having spherical tangential varieties are $C_n/P_1, C_n/P_n$.*

Proof. $\tau(C_n/P_n)$, $\tau(C_n/P_1)$ are spherical by Kac's list (using $\mathfrak{b}_0$ in the first case and $\mathfrak{b}$ in the second). We will rule out all other maximal parabolics, and once having done so, the only other possibility would be $\tau(C_n/P_{1,n})$, but this is easily eliminated by dimension considerations. (Note that Kac's list immediately implies $\tau(C_n/P_k)$ is not spherical if $k, n-k > 4$.)

In terms of matrices,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & -{}^ta & 0 \\ 0 & b_3 & 0 & -{}^tb_1 \end{pmatrix} \mid a \in \mathfrak{sl}_k,\ b_1 \in \mathfrak{sl}_{n-k}, b_2 = {}^tb_2,\ b_3 = {}^tb_3 \right\} \tag{1}$$

$$\mathfrak{b}_0 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & -{}^ta & 0 \\ 0 & b_3 & 0 & -{}^tb_1 \end{pmatrix} \mid a \in \mathfrak{b}(\mathfrak{sl}_k),\ b_1 \in \mathfrak{b}(\mathfrak{sl}_{n-k}),\ b_3 = {}^tb_3 \right\} \tag{2}$$

and

$$T_{[v_{\omega_k}]}(C_n/P_k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \\ T_2 & {}^tt' & 0 & {}^tt \\ t' & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

with $t, t' \in Mat_{k\times n-k}$, $T_2 \in S^2\mathbb{C}^k$. Note that $T_1 = t + t'$.

Fixing an initial vector with components (τ, τ', τ_2), the action of $\mathfrak{b}_0$ provides

$$\begin{aligned} t &= b_1\tau - \tau a \\ T_2 &= -(\tau_2 a + {}^t(\tau_2 a)) \\ t' &= -{}^t b_1\tau' + b_3\tau - \tau' a \end{aligned}$$

Examining the T_2 term we see both T_2 and a have dimension $\binom{k+1}{2}$ and that we may use a to exactly fill T_2. But then t may be filled only by using the lower diagonal matrix b_1, but this is not possible when $k > 1$.

Now we turn to the case of A_n. Consider $A_{n-1}/P_{k,m}$. Let $k' = m - k$ and $k'' = n - m$. Then T may be thought of as the union of three vector spaces consisting of the lower $k' \times k$, $k'' \times k$ and $k'' \times k'$ blocks. Note the naive bound for dimension reasons that

$$kk' + kk'' + k'k'' \leq \binom{k+1}{2} + \binom{k'+1}{2} + \binom{k''+1}{2} - 1.$$

Use index ranges $1 \leq i, j \leq k$, $k+1 \leq s, t \leq m$ and $m+1 \leq u, v \leq n$. Write the matrices to be filled as having elements

$$t^s_i = \tau^s_j a^j_i - a^s_t \tau^t_i \tag{4}$$

$$t^u_i = \tau^u_j a^j_i - a^u_v \tau^v_i \tag{5}$$

$$t^u_s = \tau^u_t a^t_s - a^u_v \tau^v_s \tag{6}$$

where the τ's are given (generic) constants, and given a set of t's we want to determine if there exists a set of a's that produces them. Here $a^A_B = 0$ if $A < B$, $\sum_{A=1}^n a^A_A = 0$ and otherwise the entries a^i_j, a^s_t, a^u_v of $\mathfrak{b}_0$ are independent. Now despite there being more unknowns than equations to solve in many cases, these equations are never compatible. We illustrate with the adjoint case of $\mathfrak{sl}_{n+2}$, we shift indices, having them run from 0 to $n+1$, so $1 \leq s, t \leq n$. We have

$$t^s_0 = \tau^s_0 a^0_0 - a^s_t \tau^t_0 \tag{7}$$

$$t^{n+1}_{n+1} = \tau^{n+1}_0 a^0_0 - a^{n+1}_{n+1} \tau^{n+1}_0 \tag{8}$$

$$t^{n+1}_s = \tau^{n+1}_t a^t_s - a^{n+1}_{n+1} \tau^{n+1}_s \tag{9}$$

Label the equations (7),(8),(9) respectively by $(s, 0)$, $(n+1, n+1)$ and $(0, s)$. If we consider the following linear combination of the right hand sides

$$\sum_s t^{n+1}_s (s, 0) - (\sum_s t^{n+1}_s t^s_{n+1})(n+1, n+1) - \sum_s t^s_{n+1}(0, s)$$

we get zero, which shows there are choices of t's for which there does not exist a solution. One can write out a proof of the general case similarly, but we instead include a different proof, which, while using more machinery, points out an explicit failure of sphericality.

(The following proof is best read after reading §3.)

Consider $X = A_n/P_I \subset \mathbb{P}V$ for some $I = \{i_1, \ldots, i_s\} \subset [1, n]$. Consider the desingularization of the affine cone $\hat{\tau}(X) \subset V$ given by the total space of a vector bundle $\hat{T}X$ constructed in §1.3. Continuing the notation of §1.3 with $\eta = (\hat{T}X)^*$,

$$gr(\eta) = \mathcal{O}_X(1) \oplus (\mathcal{O}_X(1) \otimes T^*X).$$

By [20] (5.1.2b) and (5.1.3.a), the normalization of the ring of coordinate functions on $\tau(X)$ is isomorphic to $H^0(G/P, Sym(\eta))$. Thus it is enough to show that for a non-maximal parabolic, that the algebra $H^0(G/P, Sym(\eta))$ is not multiplicity free.

Recall the notations $\omega_i, \epsilon_i, \alpha_i$ from [3]:

Proposition 2.6. *Assume that* $s \geq 2$. *The representation* $V^*_{2(s\omega_{i_1}+(s-1)\omega_{i_2}+\ldots+\omega_{i_s})-\epsilon_{i_1}+\epsilon_{i_s+1}}$ *occurs in* $H^0(G/P_I, S^2\eta)$ *with multiplicity* ≥ 2.

Proof. Assume that $s \geq 2$. Denote the weight indicated in the proposition by

$$\mu := 2(s\omega_{i_1} + (s-1)\omega_{i_2} + \ldots + \omega_{i_s}) - \epsilon_{i_1} + \epsilon_{i_s+1}$$

The bundle η can be filtered so the associated graded $gr(\eta)$ is the direct sum of line bundles, one for each weight space of the $\mathfrak{l}$-module M_μ. Their weights are as follows. Denote $\rho_P = \omega_{i_1} + \ldots + \omega_{i_s}$. Then the weights in $gr(\eta)$ are ρ_P and $\rho_P - \alpha$ where α runs through the positive roots not in $\mathfrak{p}$. We calculate the multiplicity of the weight μ in $S^2(gr(\eta))$. We have to count the cardinality of the set of pairs of weights in $gr(\eta)$ which add up to μ. It is the set of sums of two roots β_1, β_2 (as to choose weights $\rho_P - \beta_1$, $\rho_P - \beta_2$), not in $\mathfrak{p}$ which add up to $\epsilon_{i_1} - \epsilon_{i_s+1}$, plus one (coming from the weights ρ_P and $\rho_P - \epsilon_{i_1} + \epsilon_{i_s+1}$). This multiplicity is $i_s - i_1 + 1$. Next we notice (using Bott-Borel-Weil) that the only occurrence of the weight μ in higher cohomology are the weights giving μ in $H^1(S^2 gr(\eta))$ and the multiplicity with which it occurs is the set of pairs of roots $\beta_1 = \epsilon_{i_1} - \epsilon_j$ and $\beta_2 = \epsilon_{j+1} + \epsilon_{i_s+1}$ such that $i_1 < j \leq i_s$ is not the last element in the corresponding interval $[i_u + 1, \ldots, i_{u+1}]$. This multiplicity equals

$$(i_2 - i_1 - 1) + (i_3 - i_2 - 1) + \ldots + (i_s - i_{s-1} - 1) = i_s - i_1 - (s-1).$$

Thus the difference of multiplicities of V_μ in $H^0(S^2(gr(\eta)))$ and in $H^1(gr(\eta))$ is equal $s \geq 2$ which proves the Proposition. □

□

3. Irreducible cominuscule varieties

Let $X = G/P_{\alpha_{i_0}} \subset \mathbb{P}V_{\omega_{i_0}}$ be an irreducible rank $r \geq 3$ cominuscule variety. Continuing the notation of §1.3 with $\eta = (\hat{T}X)^*$,

$$gr(\eta) = \mathcal{O}_X(1) \oplus (\mathcal{O}_X(1) \otimes T^*X).$$

Recall that homogeneous vector bundles $E \to G/P$ correspond to $\mathfrak{p}$-modules M, where $E = G \times_P M$. In particular, the tangent bundle TX corresponds to the $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$.

Recall further that irreducible $\mathfrak{p}$-modules are in one to one correspondence with irreducible $\mathfrak{l}$-modules, which are indexed by the set of $\mathfrak{l}$-dominant weights, which we denote $\Lambda^+_{\mathfrak{l}}$. Note that $\lambda \in \Lambda^+_{\mathfrak{l}}$ iff $\lambda = a_1\omega_1 + \cdots + a_\ell\omega_\ell$, with $a_j \in \mathbb{Z}$ and $a_j \geq 0$ for $j \neq i_0$, where the ω_j are the fundamental weights of $\mathfrak{g}$ and $P = P_{\alpha_{i_0}}$.

Let M_{λ_1} denote $T^*_{[v_{\omega_{i_0}}]}X$ considered as an $\mathfrak{f} := [\mathfrak{l}, \mathfrak{l}]$-module. A uniform, for all cominuscule varieties X, decomposition of $Sym(M_{\lambda_1})$ as an $\mathfrak{f}$-module is given in [16]. These modules are exceptional in the sense of Brion [1], that is the symmetric algebra is free. The formula is

$$S^d M_{\lambda_1} = \bigoplus_{j_1+2j_2+\cdots+rj_r=d} M_{j_1\lambda_1+\ldots+j_r\lambda_r}$$

where M_{λ_2} is the complement of $M_{2\lambda_1}$ in $S^2(M_{\lambda_1})$ and $M_{\lambda_j} = S^j M_{\lambda_1} \cap (S^{j-2}M_{\lambda_1} \otimes M_{\lambda_2}) \subset M_{\lambda_1}^{\otimes j}$. In other words, consider the composition $\delta_j : S^{j-2}M_{\lambda_1} \otimes S^2 M_{\lambda_1} \to M_{\lambda_1}^{\otimes j} \to S^{j-1}M_{\lambda_1} \otimes \Lambda^2 M_{\lambda_1}$, then $M_{\lambda_j} = \ker \delta_j|_{S^{j-2}M_{\lambda_1} \otimes M_{\lambda_2}}$.

Remark 3.1. The above correspondence among $\mathfrak{f}$-modules, where generators of the symmetric algebra of M_{λ_1} under the Cartan product correspond to the prolongations of M_{λ_2}, extends to the following correspondence among $\mathfrak{l}$-modules:

The generators of the ring of covariants of $Sym(gr(\eta))$ correspond to the irreducible components of $gr(\xi)$.

This correspondence is via the projective fundamental forms. For any variety $Z \subset \mathbb{P}V$ and $z \in Z_{smooth}$, we have maps $\mathbb{F}_j : S^jT_zZ \to N_zZ$. If we let $N_k = \text{Image}\,\mathbb{F}_j$, then $gr(\xi) = \oplus_k N_k^*$, and the j-th generator of the ring of covariants of $Sym(gr(\eta))$ is ${}^t\mathbb{F}_j(N_k^*) \subset S^jT_z^*Z$. This clarifies the cryptic remarks on p. 80 of [12].

Remark 3.2. M_{λ_j} admits the geometric interpretation of the generators of the ideal of $\sigma_j(F/Q) \subset \mathbb{P}M_{\lambda_1}^*$, the variety of secant $\mathbb{P}^{j-1}$'s to $F/Q = F[v_{\lambda_1}]$, where v_{λ_1} is a highest weight vector in $M_{\lambda_1}^*$. See [16] for more information.

Here is a table of the rank r cominuscule varieties, together with a description of the F-modules M_{λ_j} and $T_{[v_{\omega_{i_0}}]}X$ as an F-module:

X	$G(k,n)$	$G_{Lag}(n,2n)$	$\mathbb{S}_{2n}$	$\mathbb{Q}^n$
G	SL_n	Sp_{2n}	$Spin_{2n}$	SO_{n+2}
F	$SL_k \times SL_{n-k}$	SL_n	SL_n	SO_n
$T_{[v_{\omega_{i_0}}]}X$	$M_{\omega_1+\omega_{n-1}} = (\mathcal{S}^* \otimes \mathcal{Q})_{[v_{\omega_{i_0}}]}$	$M_{2\omega_1} = (S^2\mathcal{S}^*)_{[v_{\omega_{i_0}}]}$	$M_{\omega_2} = (\Lambda^2\mathcal{S}^*)_{[v_{\omega_{i_0}}]}$	M_{ω_2}
M_{λ_j}	$M_{\omega_{k-j}+\omega_{k+j}} = (\Lambda^j\mathcal{S} \otimes \Lambda^j\mathcal{Q}^*)_{[v_{\omega_{i_0}}]}$	$M_{2\omega_j} = (S_{2\dots 2}\mathcal{S})_{[v_{\omega_{i_0}}]}$	$M_{\omega_{n-2j}} = (\Lambda^{2j}\mathcal{S})_{[v_{\omega_{i_0}}]}$	$\mathbb{C}$
r	$\min(k,n-k)$	n	$\llcorner \frac{n}{2} \lrcorner$	2

X	$\mathbb{OP}^2$	$G_\omega(\mathbb{O}^3,\mathbb{O}^6)$
G	E_6	E_7
F	$Spin_{10}$	E_6
$T_{[v_{\omega_{i_0}}]}X$	M_{ω_2}	M_{ω_6}
M_{λ_2}	M_{ω_6}	M_{ω_1}
M_{λ_3}	0	$\mathbb{C}$
r	2	3

We caution the reader that the λ in the M_λ are to be considered as highest weights as $\mathfrak{f}$-modules, but the labeling is as an element of the weight lattice of $\mathfrak{g}$.

Here $\mathcal{S}, \mathcal{Q}$ are respectively the tautological subspace and quotient bundles.

We almost have a description of $Sym(gr(\eta))$ from this, what is missing is the coefficient of the weights on ω_{i_0}. Because irreducible $\mathfrak{l}$-modules correspond to irreducible $\mathfrak{f}$-modules equipped with an integer weight on ω_{i_0}, adopting the notations that M_μ is the $\mathfrak{l}$-module with highest weight μ, and E_μ is the corresponding irreducible vector bundle on X, we may write

$$S^d(T^*X \otimes \mathcal{O}_X(1)) = \bigoplus_{j_1+2j_2+\cdots+rj_r=d} E_{j_1\mu_1+\ldots+j_r\mu_r}$$

where $\mu_j = \lambda_j + m_j\omega_{i_0}$ for some integers m_j, which we now determine.

Lemma 3.3. *Notations as above. $m_j = j-2$.*

Proof. There is a unique element U_{i_0} of $\mathfrak{t}$, called the *grading element* that has the property $U_{i_0}(\alpha_j) = 0$ if $j \neq i_0$ and $U_{i_0}(\alpha_{i_0}) = 1$. See e.g. [21], §3.1. In particular, TX is an eigenspace for the action of U_{i_0} with eigenvalue one, and $S^j(T^*X)$ is an eigenspace with eigenvalue $-j$ and

$S^j(T^*X \otimes \mathcal{O}(1))$ is an eigenspace with eigenvalue $-j + j(c^{-1})_{i_0,i_0}$ where c^{-1} is the inverse of the Cartan matrix. Thus

$$-j + j(c^{-1})_{i_0,i_0} = U_{i_0}(\mu_j) = U_{i_0}(\lambda_j) + m_j(c^{-1})_{i_0,i_0}$$

The lemma thus reduces to showing

$$U_{i_0}(\lambda_j) = 2(c^{-1})_{i_0,i_0} - j \tag{10}$$

which can easily be checked on a case by case basis. □

Remark 3.4. A uniform and conceptual proof of (10) is possible, but it would take us too far afield here. In particular, note that $\mu_1 = \omega_{i_0} - \alpha_{i_0}$.

Example 3.5. To verify (10) in the case of $D_n/P_n \subset \mathbb{P}V_{\omega_n}$, we have $i_0 = n$, $\lambda_j = \omega_{n-2j}$, $U_n(\omega_i) = (c^{-1})_{i,n} = \frac{i}{2}$ for $i < n-1$ and $U_n(\omega_n) = \frac{n}{4}$. We indeed have $\frac{n-2j}{2} = 2(\frac{n}{4}) - j$.

Example 3.6. To verify (10) in the case of $A_n/P_k \subset \mathbb{P}V_{\omega_k}$, we have $i_0 = k$, $\lambda_j = \omega_{k-j} + \omega_{k+j}$, $U_k(\omega_{k-j}) = \frac{(k-j)(n-k+1)}{n+1}$, $U_k(\omega_{k+j}) = \frac{k(n-(k+j)+1)}{n+1}$, $(c^{-1})_{k,k} = \frac{k(n-k+1)}{n+1}$ and we verify

$$\frac{(k-j)(n-k+1)}{n+1} + \frac{k(n-(k+j)+1)}{n+1} = 2\frac{k(n-k+1)}{n+1} - j.$$

Using that $\mu_j = \lambda_j + (j-2)\omega_{i_0}$ and $S^d(gr(\eta)) = \oplus_{k=0}^d S^k(T^*X \otimes \mathcal{O}(1)) \otimes \mathcal{O}(d-k)$, we conclude:

Proposition 3.7. *Notations as above.*

$$S^d(gr(\eta)) = \bigoplus_{a_1+2a_2+\cdots+ra_r \le d} E_{a_1\mu_1+\ldots+a_r\mu_r+(d-\sum_{j=1}^r ja_j)\omega_{i_0}}.$$

Recall that the Bott-Borel-Weil theorem implies that for irreducible homogenous vector bundles $E_\mu \to G/P$,

1. $\oplus_k H^k(E_\mu)$ is an irreducible G-module (and in particular is nonzero in at most one degree k),
2. writing $\mu = \sum_i a_i\omega_i$, if all $a_i \ge 0$ (i.e. if $\mu \in \Lambda_{\mathfrak{g}}^+$), then $H^0(E_\mu) = V_\mu^*$,
3. if all a_i but a_{i_0} are non-negative, and $a_{i_0} = -1$ then there is no cohomology,
4. if all a_i but a_{i_0} are non-negative, $a_{i_0} < -1$, and moreover $\sigma_{\alpha_{i_0}}.\mu \in \Lambda_{\mathfrak{g}}^+$ then $H^1(E_\mu) = V^*_{\sigma_{\alpha_{i_0}}.\mu}$, where V_ν is the $\mathfrak{g}$-module with highest weight ν, (Here $\sigma_{\alpha_{i_0}}.\mu$ denotes the affine action of the Weyl group, $\sigma_{\alpha_{i_0}}.\mu = \sigma_{\alpha_{i_0}}(\mu+\rho) - \rho$, where $\sigma_{\alpha_{i_0}}$ is reflection in the hyperplane orthogonal to α_{i_0}.)

Remark 3.8. One obtains the dual modules as cohomology groups above because of our convention of deleting negative root spaces to define our parabolic subalgebras.

In our case, for each E_μ that appears, the only possible negative coefficient is that of ω_{i_0}. Moreover, for $j > 1$, $\sigma_{\omega_{i_0}}.(\lambda_j) = \lambda_j$ for $j > 1$ because all the λ_j except for λ_1 are orthogonal to ω_{i_0}. (This can be verified case by case, but it is also a consequence of what is often called "Kostant's cascade".) For $j = 1$, we have $\sigma_{\alpha_{i_0}}.(a_1\lambda_1 + c\omega_{i_0}) = (a_1 + c + 1)\lambda_1 - (2 + c)\omega_{i_0}$. This last assertion follows immediately from the observation that $\mu_1 = \omega_{i_0} - \alpha_{i_0}$. In particular, if E_μ appearing in $S^d(gr(\eta))$ is neither ample, nor has no cohomology, then $\sigma_{\alpha_{i_0}}.\mu \in \Lambda_{\mathfrak{g}}^+$. In summary:

Proposition 3.9. *Recall the notations that M_μ is the $\mathfrak{l}$-module with highest weight μ, E_μ is the corresponding irreducible vector bundle on $X = G/P_{i_0} \subset \mathbb{P}V_{\omega_{i_0}}$, and V_ν is the $\mathfrak{g}$-module with* *highest weight* ν. *If* $a_1 + 2a_2 + \cdots + ra_r \leq d$, *then, letting*

$$\begin{aligned}\mu &= a_1\lambda_1 + \ldots + a_r\lambda_r + (d - 2\sum_{j=1}^{r} a_j)\omega_{i_0}\\ &= a_1\mu_1 + \ldots + a_r\mu_r + (d - \sum_{j=1}^{r} ja_j)\omega_{i_0},\end{aligned}$$

we have

(1) E_μ *is ample with* $H^0(E_\mu) = V^*_\mu$ *when* $d - 2\sum_{j=1}^r a_j \geq 0$,
(2) E_μ *has no cohomology when* $d - 2\sum_{j=1}^r a_j = -1$,
(3) E_μ *has* $H^1(E_\mu) = V^*_{\sigma_{\omega_{i_0}}.\mu}$ *when* $d - 2\sum_{j=1}^r a_j < -1$.

Note that

$$\sigma_{\omega_{i_0}}.\mu = (a_1 + d - \sum_{j=1}^{r} 2a_j + 1)\lambda_1 + a_2\lambda_2 + \ldots + a_r\lambda_r - (2 + d - 2\sum_{j=1}^{r} a_j)\omega_{i_0}.$$

Remark 3.10. Using the μ_j and ω_{i_0} has the advantage that these are the actual highest weights of the primitive $\mathfrak{l}$-modules that show up in the decomposition of $Sym(gr(\eta))$, while using the λ_j and ω_{i_0} has the advantage that all but λ_1 are orthogonal to ω_{i_0}, and (except for the case of $\mathfrak{a}_n$) all the λ_j are orthogonal to each other, and the λ_j are fundamental weights for $\mathfrak{g}$, with the exception of $\mathfrak{g} = \mathfrak{a}_n$ where they are sums of fundamental weights.

Now that we have determined the cohomology of $Sym(gr(\eta))$ we turn to $Sym(\eta)$. At most the cohomology groups appearing in $Sym(gr(\eta))$ can appear, but there can be cancellation. Note first that for a given μ, E_μ appears at most once in $Sym(gr(\eta))$. Moreover, for the E_μ appearing in $S^d(gr(\eta))$ with $H^1(E_\mu)$ nonzero, the bundle $E_{\mu'}$ with

$$\begin{aligned}\mu' &= (a_1 + (d - 2\sum_{j=1}^{r} a_j) + 1)\lambda_1 + a_2\lambda_2 + \ldots + a_r\lambda_r + (-2 - d + 2\sum_{j=1}^{r} a_j)\omega_{i_0}\\ &= (a_1 + (d - 2\sum_{j=1}^{r} a_j) + 1)\mu_1 + a_2\lambda_2 + \ldots + a_r\mu_r + (-1 - 2d - \sum_{j=1}^{r}(j-2)a_j)\omega_{i_0}\end{aligned}$$

also appears in $S^d(gr(\eta))$ and of course $H^0(E_{\mu'}) = H^1(E_\mu)$. It remains to show that these terms cancel when one passes to $H^*(Sym(\eta))$.

In order to prove that the matching terms cancel out in the spectral sequence we use the technique of [19]. The essential point is that

$$Ext^1(\mathcal{O}(1), T^*X \otimes \mathcal{O}(1)) = H^1(Hom(\mathcal{O}(1), T^*X \otimes \mathcal{O}(1)))$$

(see [5], proposition 6.5), and $Hom(\mathcal{O}(1), T^*X \otimes \mathcal{O}(1)) \simeq T^*X$. Now $T^*X = E_{\lambda_1 - 2\omega_{i_0}}$ and applying the Bott-Borel-Weil theorem again, we see $H^1(E_{\lambda_1 - 2\omega_{i_0}}) = \mathbb{C}$. Thus there is a unique (up to scale) nontrivial extension.

The quiver representation of a quiver $\mathcal{Q}_X$ defined in [19] corresponding to $S^d\eta$ has one dimensional spaces attached to vertices with the highest weights of L-modules $M_{(d-\sum ja_j)\omega_{i_0}+a_1\mu_1+\ldots+a_r\mu_r}$ for $a_1 + 2a_2 + \cdots + ra_r \leq d$. The arrows connect the weight of M_μ to that of $M_{\mu'}$ where μ, μ' are as above. Then, noting that η is indeed a nontrivial extension because it is acted on nontrivially by $\mathfrak{n}$, [19], Proposition 6.7 assures that the connecting homomorphism between two cancelling terms is nonzero.

Recall that for an algebra $\mathcal{A}$ that has the structure of a $\mathfrak{g}$-module, the *ring of covariants* of $\mathcal{A}$ is the set of elements of $\mathcal{A}$ annihilated by all positive root vectors in $\mathfrak{g}$ (or, if working with a corresponding algebraic group G, the elements invariant under the action of a unipotent radical of G). Another perspective is that the ring of covariants is the generators of $\mathcal{A}$ as an algebra with the multiplication by Cartan product instead of its usual multiplication.

Theorem 3.11. *Let $X = G/P_{i_0} \subset \mathbb{P}V_{\omega_{i_0}}$ be rank $r \geq 3$ cominuscule variety. Let $K[\tau(X)]$ denote the homogeneous coordinate ring of $\tau(X)$. Then, continuing the notation of above*

(1)

$$K[\tau(X)]_d = \bigoplus_{2\sum_{j=1}^{r} a_j \leq \ min\ \{d, \sum_{j=1}^{r} ja_j\}} V^*_{(d-\sum_j ja_j)\omega_{i_0}+a_1\mu_1+\ldots+a_r\mu_r}$$

(2) *The ring of covariants of $K[\tau(X)]$ is generated by the modules $V^*_{\omega_{i_0}}$, $V^*_{i\mu_1+\mu_s}$ with $1 \leq i \leq s-2$, and $3 \leq s \leq r$, and $V^*_{\mu_2}, \ldots, V^*_{\mu_r}$. Thus the ring is generated in degrees $\leq 2(r-1)$.*

Remark 3.12. The marked Dynkin diagram describing the module V^*_μ is the marked Dynkin diagram of V_μ reflected by the $\mathbb{Z}_2$-symmetry of the diagram.

Proof. To prove the first assertion, by Theorem 1.4.2 and the preceeding paragraph, we just need to calculate $H^0(S^d(\eta))$. By Proposition 3.9, a module $V^*_{(d-\sum_j ja_j)\omega_{i_0}+a_1\mu_1+\ldots+a_r\mu_r}$ is in $H^0(S^d(gr(\eta))$ if $2\sum a_j \leq d$, and by the discussion above, to obtain $H^0(S^d(\eta))$ we must subtract the modules V^*_μ such that $(\sigma_{\alpha_{i_0}}.)^{-1}\mu$ also occurs in $S^d(gr(\eta))$. Since $(\sigma_{\alpha_{i_0}}.)^{-1}(a\lambda_1 + c\omega_{i_0}) = (a+c+1)\lambda_1 + (-2-c)\omega_{i_0}$ we need to subtract the modules with

$$(a_1 + (d - 2\sum_j a_j) + 1) + 2a_2 + \cdots + ra_r \leq d,$$

i.e., we require $(a_1 - 2\sum_j a_j + 1) + 2a_2 + \cdots + ra_r > 0$, i.e., $2\sum a_j \leq \sum ja_j$.

For the second assertion, it is clear that $V^*_{\omega_{i_0}}$ is among the generators. To prove that the other generators are as described in the statement, we use induction on $s :=\ \min\ \{i \mid i > 1, a_i > 0\}$. Consider an r-tuple $a = (a_1, ..., a_r)$ such that $a_2, ..., a_{s-1} = 0$ satisfying $2\sum a_j \leq ja_j$. Set $k =\ \min\ (a_s, \llcorner \frac{a_1}{s-2} \lrcorner)$ and subtract $k(s-2, 0, ..., 0, 1, 0, ...,)$. Either one obtains $a_s = 0$ and we may go to the next step of the induction or $a_s > 0$ and $a_1 < s-2$. But such a vector a is a non-negative linear combination of vectors $(a_1, 0, ..., 0, 1, 0, ...,)$ and the vectors in the basis for larger s. □

To determine generators of the ideal we must calculate the modules $H^d(\Lambda^{d+1}\xi)$. Here $\xi = N^*_X(1)$. Unfortunately even decomposing $\Lambda^i M_{\lambda_j}$ in general is a difficult problem, which is why we are only able to determine explicit generating modules in a few special cases.

4. Multiple embeddings

In this section we generalize the results of §3 to multiple embeddings of X. The results become easier, because all higher cohomology of $Sym(gr(\eta))$ vanishes. Assume that X is embedded into $V^*_{N\omega_{i_0}}$ by the N-tuple embedding with $N \geq 2$.

Using the notation of §1.3, with $\eta = (\hat{T}X)^*$,

$$gr(\eta) = \mathcal{O}_X(N) \oplus (\mathcal{O}_X(N) \otimes T^*X).$$

This implies

Proposition 4.1. *Notations as above.*

$$S^d(gr(\eta)) = \bigoplus_{a_1+2a_2+\cdots+ra_r \leq d} E_{a_1\mu_1+\ldots+a_r\mu_r+(Nd-\sum_{j=1}^{r} ja_j)\omega_{i_0}}.$$

By repeating the reasoning from the previous section we get

Theorem 4.2. *Let $X = G/P_{i_0} \subset \mathbb{P}V_{N\omega_{i_0}}$ be a irreducible rank $r \geq 3$ generalized cominuscule variety, embedded by an N-tuple embedding with $N \geq 2$. Let $K[\tau(X)]$ denote the homogeneous coordinate ring of $\tau(X)$. Then, continuing the notation of above*

(1)
$$K[\tau(X)]_d = \bigoplus_{\sum_j ja_j \leq d} V^*_{(Nd-\sum_j ja_j)\omega_{i_0}+a_1\mu_1+\ldots+a_r\mu_r}$$

(2) *The ring of covariants of $K[\tau(X)]$ is generated by the modules $V^*_{N\omega_{i_0}}$, and $V^*_{(N-1)\omega_{i_0}+\mu_j}$, for $1 \leq j \leq r$. Thus the ring is generated in degree 1. However, since in degree 1 we have more than one representation, the embedding of $\tau(X)$ into $V^*_{\omega_{i_0}}$ is not linearly normal.*

Proof. The result follows at once by observing that all bundles $E_{a_1\mu_1+\ldots+a_r\mu_r+(Nd-\sum_{j=1}^r ja_j)\omega_{i_0}}$ are now ample. □

Corollary 4.3. *Let $X \subset \mathbb{P}V$ be a rank ≥ 3 cominuscule variety. Then $\tau(X)$ is not quadratically normal.*

5. Reducible cominuscule varieties

Let $X = Seg(X_1 \times \cdots \times X_m) \subset \mathbb{P}V = \mathbb{P}(W_{\omega^1_{i_0}} \otimes \cdots \otimes W_{\omega^m_{i_0}})$ where each $X_i = G^i/P_{\alpha^i_{i_0}} \subset \mathbb{P}W_{\omega^i_{i_0}}$ is a rank r_i cominuscule variety so the rank of X is $r := r_1+\cdots+r_m$. (We leave the case of non-minimally embedded factors to the reader.) Write $\mathcal{O}_X(p_1,...,p_m) = \mathcal{O}_{X_1}(p_1) \otimes \cdots \otimes \mathcal{O}_{X_m}(p_m)$ where we have omitted the pullback maps from the notation. Then

$$gr(\hat{T}X) = \mathcal{O}_X(-1,...,-1) \oplus \oplus_{j=1}^m \mathcal{O}_X(-1,...,-1) \otimes TX_j.$$

We adopt the notation ${}^sM_{\lambda^s_j} = M_{\lambda^s_j}(X_s)$ and $\eta_s = \eta(X_s)$ following the notation of §3 for the irreducible cases, in particular $a^s_{j_s}$ corresponds to a_{j_s} of the s-th factor and similarly for $\omega^s_{i_0}$, μ^s_j etc... Then

$$S^d(gr(\eta)) = \bigoplus_{p_1+\cdots+p_m \leq d} \mathcal{O}_X(d-p_1,...,d-p_m) \otimes S^{p_1}(gr(\eta_1)) \otimes \cdots \otimes S^{p_m}(gr(\eta_m)).$$

Lemma 5.1. *Notations as above.*

(1) *$H^k(S^d(gr(\eta)) = 0$ for $k > 1$.*
(2) *The modules of the form $\otimes_s {}^sM_{\mu^s}$ are in the positive cone of the Weyl chamber are those whose s-th component is in the positive cone for each s.*
(3) *The modules appearing in $S^d(gr(\eta))$ with no H^0 term are those where $d < 2\sum_{j=1}^{r_s} a^s_j$ for some $s \in \{1,...,m\}$. Note that this can occur for at most one such s and that such a term will contribute a module to H^1.*
(4) *The modules appearing in $H^1(gr(Sym(\eta)))$ all appear in $H^0(gr(Sym(\eta)))$ with the same multiplicity. These terms cancel when one passes to $H^*(Sym(\eta))$.*

Lemma 5.1 follows from proposition 3.9 by observing that the Weyl group acts independently on each factor and at most one factor can fail to be ample.

Theorem 5.2. *Let $X = Seg(X_1 \times \cdots \times X_m) \subset \mathbb{P}V = \mathbb{P}(W_{\omega^1_{i_0}} \otimes \cdots \otimes W_{\omega^m_{i_0}})$ be a homogeneously embedded rank $r = r_1 + \cdots + r_m$ cominuscule variety with $r \geq 3$. Let $K[\tau(X)]$ denote the homogeneous coordinate ring of $\tau(X)$. Then*

(1) *Let $p_s = a^s_1 + 2a^s_2 + \ldots + r_s a^s_{r_s}$.*

$$K[\tau(X)]_d = {\bigoplus}' \bigotimes_{s=1}^{m} {}^sV^*_{(d-\sum_{j=1}^{r_s} ja^s_j)\omega^s_{i_0}+a^s_1\mu^s_1+\cdots+a^s_{r_s})}.$$

The sum $\oplus'$ *is over sets* (a^s_j) *such that* $\forall s,\ 2\sum_{j=1}^{r_s} a^s_j \leq\ min\ \{d, \sum_{j=1}^{r_s} ja^s_j + p_1 + \cdots + \hat{p}_s + \cdots + p_m\}$ *and* $p_1 + \cdots + p_m \leq d$.

(2) *The generators, which (aside from those of type (i.)) we label by the sets of integers* a^s_j *come in four types:*

(i.) Fix s *and for each* $e \geq 0$ *take the generators in degree* e *of* $K[\tau(X_s)]$ *on the* s*-th coordinate (listed in Theorem 3.11) tensored with the representation* ${}^tV^*_{e\omega_{i_0}}$ *on the remaining coordinates,*

(ii.) Fix $s_1, s_2 \in \{1, ..., m\}$ *and* $j_2 \in \{1, ..., r_{s_2}\}$*, we have* $j_2 \geq a^{s_1}_1 > 0$*,* $a^{s_1}_\rho = 0$ *for* $\rho > 1$*,* $a^t_j = 0$ *for all* $t \neq s_1$ *except for* $a^{s_2}_{j_2} = 1$*.*

(iii.)Fix s_1, s_2*,* $a^{s_1}_1 = a^{s_2}_1 = 1$ *and all other* a^t_j *are zero.*

(iv.) Fix s_1, s_2, s_3*,* $a^{s_1}_1 = a^{s_2}_1 = a^{s_3}_1 = 1$ *and all other* a^t_j *are zero.*

Thus $K[\tau(X)]$ *is generated in degrees up to* $max\ \{3, 2r_0\}$ *where* $r_0 = max_s r_s$*.*

The proof is similar to the irreducible case.

Example 5.3 (Segre varieties)**.** Consider m factors of $\mathbb{P}^1$, $Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1) \subset \mathbb{P}(K^2 \otimes \cdots \otimes K^2)$. The representations occurring in $K[\tau(X)]_d$ are all modules that are tensor products of Schur functors with partitions of d of length at most two. For each $s = 1, \ldots, m$ the weight $\omega^s_{i_0}$ corresponds to the partition $(1, 0)$ and the weight λ^s_1 corresponds to $(0, 1)$. The generators of $K[\tau(X)]$ are as follows. There is a representation with the weight $\otimes_{s=1}^m \lambda^s_0 = \otimes_{s=1}^m (1, 0)$ in degree one, and the representations with the weights ν_I for any $I \subset [1, ..., n]$ $|I| = 2$ or 3, where for $|I| = i$ we have $(\nu_I)_s = (i - 1, 1)$ for $s \in I$ and $(\nu_I)_s = (i, 0)$ for $s \notin I$. Since the rank of each X_i is one, we have that all generators of $K[\tau(X)]$ occur in degree at most three. It will follow from the results of §6 that $I(\tau(X))$ is generated in degrees at most six.

6. The degeneration argument

We use the notation of §15 of [8]. Since in [8] algebraic groups are used instead of Lie algebras when discussing weights etc..., for this section only we use Λ_G instead of $\Lambda_{\mathfrak{g}}$, although since we are in characteristic zero, we could have just as well used Lie algebras. Let G be a linearly reductive group, T a maximal torus and U a unipotent radical. For an algebra A with rational G-action Grosshans (Lemma 15.1) constructs a homomorphism $h : \Lambda_G \to \mathbf{Z}$ satisfying the properties

- a) $h(\omega) \in \mathbb{Z}_{\geq 0}$ when $\omega \in \Lambda^+_G$,
- b) if $\chi > \chi'$(i.e. the difference is a sum of positive roots), then $h(\chi) > h(\chi')$,
- c) $h(g_j\chi) = h(\chi)$ where $\{g_j\}$ is the set of representatives of cosets of G with respect to the connected component of the identity G^0 (this is trivial for connected G),

For an algebra A with a rational G-action Grosshans defines

$$A_n = \{a \in A \mid h(\chi) \leq n \text{ for all weights } \chi \text{ of } T \text{ in the span } \langle G.a\rangle\}.$$

We define

$$gr(A) := \oplus_{n\geq 0}(A_n/A_{n-1}).$$

This is a commutative algebra with a rational G-action, with the product induced by the product in A. The algebras A and $gr(A)$ have the same algebras of U-invariants. Define

$$D := \sum_{n\geq 0} A_n x^n \subset A[x].$$

The algebra D has a rational G-action and it has the following properties:

- d) $D/xD = gr(A)$,
- e) $D[\frac{1}{x}] = A[x, \frac{1}{x}]$.

Theorem 15.14 in [8] implies:

Theorem 6.1. *Let $i : K[x] \to D$ be an inclusion. Then D is flat over $K[x]$. The fiber of i over a maximal ideal $(x-\alpha)$, $\alpha \in K^*$, is isomorphic to A, and the fiber over (x) is isomorphic to $gr(A)$.*

Another way to think about $gr(A)$ is that it is A with the product deformed so only the Cartan piece of the product of A is retained.

We apply the theorem to multiplicity free algebras.

Theorem 6.2. *Assume that A is multiplicity free graded domain with rational G-action preserving the grading. Assume $gr(A)$ is generated by representations of degree $\leq d$. Then the defining ideal of A is generated in degrees $\leq 2d$.*

Proof. Let $\Theta := \{\lambda \in \Lambda_G^+ \mid V_\lambda \subset A\}$. Note that A is multiplicity free and a domain, so $V_\lambda, V_\mu \subset A$ implies $V_{\lambda+\mu} \subset A$ thus Θ is an abelian sub-semi-group of Λ_G^+. Consider $gr(A)$. This is an algebra that additively is

$$gr(A) = \oplus_{\lambda \in \Theta} V_\lambda$$

with the product given by Cartan product. By the previous theorem $gr(A)$ is a special fiber of a flat deformation with general fiber A. Introduce a new degree on $gr(A)$ by setting the degrees of the generators of Θ to one. Now by an unpublished theorem of Kostant [7] $gr(A)$ has relations in degrees ≤ 2. This means the original degrees of these relations are $\leq 2d$. But there is a presentation of the general fiber given by the generators and relations in the same degrees as that of a special fiber. This proves our statement. □

7. Further information on the ideals

First note that $\tau(X) \subseteq \sigma(X)$ and, as discussed in [17], in most cases there is a *subspace variety* or *rank variety* containing $\sigma(X)$. Thus we may study the equations of $\tau(X)$ by first understanding certain "primitive" cases and then the ideals of the rank varieties themselves. The ideals of rank varieties are possible to determine by the method of [20] in many cases. Assuming we have both a set of generators of the ideal of a primitive case and of the relevant rank varieties, one must still determine which generators of the ideal of the rank variety become redundant when considered as members of the ideal of $\tau(X)$.

7.1. **Grassmannians.** Consider $G(r, N)$. Write $V^* = \mathbb{C}^N$. In this case the primitive varieties are $G(r, 2r)$ and the relevant rank variety in $\Lambda^r V^*$ is

$$Z_{r,N} = \{T \in \Lambda^r V^* \mid \exists W \subset V^*, \dim W = 2r,\ T \in \Lambda^r W\}.$$

These rank varieties are discussed in §7.3 of [20].

Consider $G(3, N)$. Here the primitive case is $G(3, 6)$ and $\tau(G(3, 6))$ is a quartic hypersurface whose equation is the unique occurance of the trivial representation of $\mathfrak{sl}_6$ in $S^4(\Lambda^3\mathbb{C}^6)$ corresponding to the partition $S_{222222}(\mathbb{C}^6) = S_{2^6}(\mathbb{C}^6)$. Thus the ideal of $\tau(G(3, V^*))$ is spanned by $S_{2^6}V$ and the generators of the ideal of the subspace varieties. For $G(3, 7)$ the ideal of the variety of tensors of rank 6 is generated by $S_{3111111}V$. For $G(3, 8)$ the variety of tensors of rank ≤ 7 is generated by $S_{3,2^2,1^5}V$ but in [18] we show that this is in the ideal generated by $S_{2^6}V$ and $S_{3,1^6}V$. (See [18] for proofs of the assertions regarding the generators of these rank varieties.)

Theorem 7.1. *The ideal of $\tau(G(3, V^*)) \subset \mathbb{P}(\Lambda^3 V^*)$ is generated by $S_{2^6}V$ in degree four and $S_{3,1^6}V$ in degree three.*

For $G(4, 8)$ we calculated the Euler characteristics of the bundles $\Lambda^{d+1}\xi$ directly. In small degrees we can recover $H^d(\Lambda^{d+1}\xi)$ from the Euler characteristic to prove:

Theorem 7.2. *The ideal of* $\tau(G(4,8))$ *has, among its generators,* $S_{1^8}V$ *in degree two,* $S_{2^6}V$ *and* $S_{3^2,1^6}V$ *in degree three and* $S_{4,2^6}V$ *in degree four.*

We expect that these modules in fact generate the ideal.

For $r > 4$ the calculation becomes more difficult.

7.2. **Legendrian varieties.** These are the cases where $X \subset \mathbb{P}V$ is $v_3(\mathbb{P}^1)$, $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, $G_{Lag}(3,6) = C_3/P_3$, $G(3,6)$, $\mathbb{S}_6 = D_6/P_6$, E_7/P_7, and $Seg(\mathbb{P}^1 \times Q)$ where Q is a quadric hypersurface. Here $\tau(X)$ is a quartic hypersurface. The ring of covariants is free and has a uniform description: it is generated by V in degree one, $\mathfrak{g}$ in degree two, V in degree three and V_2 in degree four, where $V_2 \subset \Lambda^2 V$ is the complement of the line spanned by the symplectic form. See [16] for more details.

7.3. **Scorza varieties.** The homogeneous varieties X with $\tau(X)$ degenerate (i.e., of dimension less than $2\dim X$) coincide exactly with the rank 2 cominiscule varieties: Segre varieties, $Seg(\mathbb{P}^a \times \mathbb{P}^b) \subset \mathbb{P}(K^{a+1} \otimes K^{b+1})$, Grassmannians of two-planes $G(2,n) \subset \mathbb{P}(\Lambda^2 K^n)$, the Cayley plane $E_6/P_6 \subset \mathbb{P}^{26}$, quadric hypersurfaces and the spinor variety D_5/P_5 (a component of the Grassmannian of 5-planes in K^{10} isotropic for a quadratic form). The ideals of the tangential varieties in these cases are either empty (when $\tau(X) = \mathbb{P}V$) or are generated by cubics.

In all these cases the secant variety of X coincides with the tangential variety. These cases also have a uniform description as rank one elements in Jordan algebras with the tangential varieties as the rank at most two elements. The ideal of $\tau(X)$ is generated in degree three by the three by three minors in the Jordan algebra, the algebra of covariants of $K[\tau(X)]$ is free and generated by the minors of the Jordan algebra of various sizes (other than three by three). See [16] for more details.

7.4. **Spinor varieties.** For D_4, D_5 the tangential varieties $\tau(D_n/P_n)$ are the ambient spaces, and these modules are exceptional in the sense of Brion [1], in that the ring of covariants is free. The generators are respectively V_{ω_3} in degree one and the trivial representation in degree two for $n = 4$ and V_{ω_4} in degree one and V_{ω_1} in degree two for $n = 5$.

For $n = 6$ we are in the Legendrian case.

For $n = 7$, checking the generators of the lattice of weights occurring in $K[\tau(X)]$ we see that the ring of covariants is still free.

For $n \geq 8$ it is no longer free. In fact for $n = 8$ among the generators of $K[\tau(X)]$ are $V_{\omega_2+\omega_7}$ in degree three, $V_{\omega_6+\omega_7}$ in degree five $V_{\omega_2+\omega_4}$ in degree three and $V_{2\omega_7}$ in degree four. The Cartan products of the first two and last two representations are the same.

7.5. **Segre varieties.** We collect all results about the tangential varieties of the Segre embeddings of products of projective spaces. We use the geometric technique and the deformation argument from §6. We start with the case of products of projective lines.

Theorem 7.3. *Let* $X = Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_m^*) \subset \mathbb{P}(A_1^* \otimes \cdots \otimes A_m^*)$ *where* A_j *is a vector space of dimension* 2 *for* $j = 1, \ldots, m$. *Then*

1. *The ideal of* $\tau(X)$ *is generated in degree at most* 6.
2. *The last term in the minimal free resolution of* $K[\tau(X)]$ *is* $\otimes_{j=1}^m S_{(2^{m-1}-2,2^{m-1}-m+1)}A_j$.

Proof. The first statement follows from Example 5.3 and from Theorem 6.2. The second statement follows from Theorem 5.1.2 in [20]. The top term in the resolution is easily seen to come from the cohomology of top exterior power of ξ. Then the result follows from Serre's theorem on cohomology of line bundles on projective space. □

Now we pass to the general case where $X = Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_m^*) \subset \mathbb{P}(A_1^* \otimes \cdots \otimes A_m^*)$ with $\dim A_j = a_j + 1$ for $j = 1, \ldots, r$.

Theorem 7.4. *For Segre varieties $X = Seg(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$, the tangential variety $\tau(X)$ is arithmetically Cohen-Macaulay.*

Proof. In the following proof we use a relative version of the machinery in [20]. We will be terse here because a very similar argument with more detail is in [17], §5.

Let $Sub_{r,\dots,r}$ denote the rank or subspace variety whose desingularization is given by the rank 2^m tautological subspace bundle $\mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_m \to \Pi_{j=1}^m G(2, A_j^*) = B$ (see [17], §3). Note that $\dim Sub_{r,\dots,r} = 2^m + \sum_{j=1}^m 2(a_j - 1)$. We have $\tau(X) \subset Sub_{r,\dots,r}$ (as we even have the secant variety of X contained in Y, see e.g., [17]).

The variety $Sub_{r,\dots,r}$ has a desingularization that allows one to apply the geometric technique from [20]. In the notation of Theorem 1.4, we take $\eta = \mathcal{R}_1^* \otimes \cdots \otimes \mathcal{R}_m^*$ and $\xi := Ker((A_1 \otimes \dots \otimes A_m \otimes \mathcal{O}_B) \to \eta)$.

We consider the sheaf of algebras $\mathcal{B} := Sym(\eta)$. We show the hypotheses of Theorem 1.4 are satisfied. We need the following lemma from [17]:

Lemma 7.5. *[[17], Lemma 5.2] Let $\pi_j = (p_{j,1}, \dots, p_{j,r})$ be partitions. Consider the sheaf*

$$\mathcal{M} := \otimes_{j=1}^m S_{\pi_j} \mathcal{R}_j^* \otimes \mathcal{B}.$$

(1) *Assume that $p_{j,1} \geq -a_j + 1$ for $1 \leq j \leq m$. Then $\mathcal{M}$ is acyclic.*
(2) *Assume that $p_{j,1} \geq 0$ and $p_{j,1} \leq r^{m-1} - r$ for $1 \leq j \leq m$. Then the $Sym(A_1 \otimes \cdots \otimes A_m)$-module $H^0(B, \mathcal{M})$, which is supported in $Sub_{r,\dots,r}$, is a maximal Cohen-Macaulay module.*

Now we use the desingularization of $\hat{\tau}(X)$ by $\hat{T}X$. It is a vector bundle of rank $m+1$ which is a factor of the bundle $\mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_m$ defining the desingularization of $Sub_{r,\dots,r}$. To estimate higher direct images we first analyze the finite free resolution of $\hat{\tau}(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$ (m copies) in the relative setting (taking $\mathcal{R}_j$ instead A_j^*).

We apply the above lemma to this resolution. By Theorem 7.4,(2) each term in this resolution satisfies conditions (1) and (2) of Lemma 7.5. Thus each term has no higher direct images and its sections form a maximal Cohen-Macaulay module supported in Y. This proves the vanishing of higher direct images of the structure sheaf of Z and thus proves the rational singularities. Taking resolution of each term in that complex and using an iterated mapping cone construction we get a nonminimal resolution of the coordinate ring of tangential variety whose length equal its codimension. This resolution implies that the coordinate ring $K[\tau(X)]$ is Cohen-Macaulay. □

Calculating the Euler characteristic of $\Lambda^{d+1} gr(\xi)$ in low degrees we uncover certain generators of $I(\tau(X))$ which we expect to generate the ideal.

Conjecture 7.6. *$I(\tau(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*))$ is generated the quadrics in $S^2(A_1 \otimes \cdots \otimes A_m)$ which have at least four Λ^2 factors, the cubics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$.*

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