323 Honors homework due 4/15

April 5, 2014

Let $\mathfrak{S}_n$ denote the set of permutations of $\{1, \ldots, n\}$. This set is a *group*, i.e., there is a well defined multiplication: given $\sigma, \tau \in \mathfrak{S}_n$, the composition $\sigma \circ \tau$ (performing τ then σ on a set) is also a permutation, every permutation has an inverse permutation and there is an identity permutation. For each $\sigma \in \mathfrak{S}_n$, let $P_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ denote the corresponding permutation matrix.

1. There is a special line in $\mathbb{R}^n$ that is an eigenline for every P_σ. Find the line and the corresponding eigenvalue.

2. Show that every eigenvalue λ of a permutation matrix P_σ is such that $|\lambda| := \lambda\bar{\lambda} = 1$.

3. Find the eigenvalues of the permutation $(1, 2, \ldots, n)$ (we did this in class).

4. I mentioned in class that every permutation σ can be decomposed into a product of disjoint cycles. Say those cycles have lengths $p_1, p_2, \ldots, p_t$ where if an element is fixed by σ it gets its own cycle of length one, a transposition, e.g. $(1, 2)$ has length two etc... Order the cycles such that $p_1 \geq p_2 \geq \cdots \geq p_t$. Note that we must have $p_1 + p_2 + \cdots + p_t = n$ as every element of $\{1, \ldots, n\}$ must be accounted for. Determine the eigenvalues of P_σ in terms of the integers $p_1, \ldots, p_t$.

5. Note that the previous exercise implies that any two permutations with the same $p_1, \ldots, p_t$ have the same eigenvalues. We also saw that two non-defective matrices with the same eigenvalues are conjugate to one another (recall that $A = XBX^{-1}$ for some X implies A, B have the same eigenvalues). Show that two permutations have the same $p_1, \ldots, p_t$ if and only if they are conjugate.