## 323 HONORS SPRING 2014, HOMEWORK DUE 2/18

### 1. How to recover a polynomial from samples

**Problem** : Let $P(x)$ be a polynomial of degree at most $n$, presented in some indirect or inefficient manner, such that our computer can, for any (say rational) number $\lambda$, compute $P(\lambda)$ for us. How can we obtain an expression of $P(x)$ of the form $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$ for some constants $a_j$?

**Answer** : Sample $P$ at $n+1$ points $\lambda_0, \cdots, \lambda_n$ (e.g., in practice you could take these to be $0, 1, 2, \cdots, n$). We obtain $n+1$ linear equations for the $n+1$ unknowns $a_0, \cdots, a_n$. Explicitly:

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^n \\ & & \vdots & & \\ 1 & \lambda_n & \lambda_0^2 & \cdots & \lambda_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} P(\lambda_0) \\ \vdots \\ P(\lambda_n) \end{pmatrix} \tag{1}$$

So taking the inverse of this size $(n+1)$ Vandermonde matrix we obtain the coefficients.

*Remark* 1.1. Note that if we wanted to do this calculation for a collection of polynomials, after the initial overhead of computing once and for all the inverse of a size $(n+1)$ Vandermonde matrix of our choosing, it is very cheap computationally to recover the coefficients of the polynomial. We will see several applications taking advantage of this.

**Exercise 1.2:** Find the cofficients of the degree three polynomial $p(x)$ such that $p(0) = 1, p(1) = 7, p(2) = 23, p(3) = 61$.

### 2. A fast algorithm for evaluating elementary symmetric functions

Write

$$e_k(x) = e_k(x_1, \cdots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \tag{2}$$

for the $k$-th elementary symmetric function in $n$ variables. Note that this expression involves $\binom{n}{k}$ monomials, so to evaluate it on a given $n$-tuple of numbers in this form involves a large number of arithmetic operations (think of the case $k = \lfloor \frac{n}{2} \rfloor$ and recall that $\binom{2m}{m} \simeq \frac{4^m}{\sqrt{\pi m}}$).

The first step will be to note the following polynomial $E(x,t) := \Pi_{i=1}^n (x_i + t)$ is such that the coefficient of $t^k$ is $e_{n-k}(x)$, where by convention we set $e_0(x) = 1$. (Such a function is called a *generating function*.)

**Exercise 2.1:** Show that the Taylor series expansion of $P(x,t) := \sum_{i=1}^n \frac{x_i}{1 - t x_i}$ is a generating function for the power sum functions $p_k(x) := x_1^k + \cdots + x_n^k$.

We think of $E(x,t)$ as a polynomial in $t$ whose coefficients we would like to recover. Write

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^n \\ & & \vdots & & \\ 1 & \lambda_n & \lambda_0^2 & \cdots & \lambda_n^n \end{pmatrix} \begin{pmatrix} e_0 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} E(x,\lambda_0) \\ \vdots \\ E(x,\lambda_n) \end{pmatrix} \tag{3}$$

Note that each $E(x, \lambda_i) = (x_1 - \lambda_i) \cdots (x_n - \lambda_i)$ is just a product of $n$ linear forms and hence is easy to evaluate on any given $x$. (More precisely, one only needs to perform $n$ additions and

$n-1$ multiplications.) Thus, even $e_{\lfloor \frac{n}{2} \rfloor}$ can be evaluated using just on the order of $n^2$ arithmetic operations compared with $O(2^n)$ using the naive formula.

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