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Let $\mathfrak{S}_n$ denote the set (group) of permutations of $\{1, \cdots, n\}$. We saw that for each $\sigma \in \mathfrak{S}_n$ we get a linear map $P_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ given by $P_\sigma(e_j) = e_{\sigma(j)}$. We also saw that there is a common eigenvector for all $\sigma \in \mathfrak{S}_n$, namely $e_1 + \cdots + e_n$. Let $L \subset \mathbb{R}^n$ be the corresponding eigenline. We also saw that the matrices P_σ are orthogonal with respect to the standard inner product on $\mathbb{R}^n$, so $\langle P_\sigma v, P_\sigma w\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$.

(1) Let $L^\perp \subset \mathbb{R}^n$ be the orthogonal complement to L. Show that for all $\sigma \in \mathfrak{S}_n$, and $u \in L^\perp$, that $P_\sigma u \in L^\perp$. (One says that $\mathfrak{S}_n$ *preserves* $L^\perp$.) Hint: show that $e_1 - e_2, e_1 - e_3, \cdots, e_1 - e_n$ is a basis of $L^\perp$ and compute with this basis.

(2) Let $\mathbb{R}^n \otimes \mathbb{R}^n$ denote the space of bilinear maps $\mathbb{R}^{n*} \times \mathbb{R}^{n*} \to \mathbb{R}$. Since we have the standard inner product on $\mathbb{R}^n$, this may be identified with the space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and we will do so. In bases, given an $n \times n$ matrix A one obtains the bilinear map B_A where $B_A(v, w) = v^T A w$. Show that if A is a symmetric matrix then $B_A(v, w) = B_A(w, v)$ for all $v, w \in \mathbb{R}^n$ and if A is a skew-symmetric matrix (i.e., $A^T = -A$) that $B_A(v, w) = -B_A(w, v)$ for all $v, w \in \mathbb{R}^n$.

(3) Given $\sigma \in \mathfrak{S}_n$ define a linear map $P_\sigma^{\otimes 2} : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$, given by $(P_\sigma^{\otimes 2} B)(v, w) = B(P_\sigma{}^{-1} v, P_\sigma{}^{-1} w)$. Show that in matrices this is $P_\sigma^{\otimes 2}(A) = P_\sigma A P_\sigma{}^{-1}$ and that for permutations σ, τ, on has $P_{\sigma\circ\tau}^{\otimes 2} = P_\sigma^{\otimes 2} P_\tau^{\otimes 2}$. (Here $\sigma \circ \tau$ denotes the composition of permutations.) This latter equality explains why one inserts the inverse in the definition of $P_\sigma^{\otimes 2}$: so composition of linear maps is compatible with composition of permutations.

(4) Recall that any $n \times n$ matrix A can be uniquely written as the sum of a symmetric matrix plus a skew symmetric matrix. Write $S^2\mathbb{R}^n$ and $\Lambda^2\mathbb{R}^n$ respectively for the spaces of symmetric and skew-symmetric matrices. So in other words, $\mathbb{R}^n \otimes \mathbb{R}^n = S^2\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n$, where $S^2\mathbb{R}^n$ is the space of symmetric bilinear maps and $\Lambda^2\mathbb{R}^n$ is the space of skew-symmetric bilinear maps.

Show that $\mathfrak{S}_n$ preserves this decomposition, i.e., for all $A \in S^2\mathbb{R}^n$, $P_\sigma^{\otimes 2}(A) \in S^2\mathbb{R}^n$ and for all $A \in \Lambda^2\mathbb{R}^n$, $P_\sigma^{\otimes 2}(A) \in \Lambda^2\mathbb{R}^n$.

(5) We can refine the above decomposition by using the decomposition $\mathbb{R}^n = L \oplus L^\perp$, to obtain $\Lambda^2\mathbb{R}^n = \Lambda^2 L^\perp \oplus (L^\perp \otimes L)$ and $S^2\mathbb{R}^n = S^2 L^\perp \oplus (L^\perp \otimes L) \oplus S^2 L$. Show that $\mathfrak{S}_n$ also preserves this refined decomposition. (Recall that for vector spaces A, B with inner products, $A \otimes B$ is the space of the bilinear maps $A \times B \to \mathbb{R}$.) Hints: for $x, y \in \mathbb{R}^n$, the matrix of $B_{x\otimes y}$ is xy^T. Show that $B_{x\otimes y - y\otimes x} \in \Lambda^2\mathbb{R}^n$ and $B_{x\otimes y + y\otimes x} \in S^2\mathbb{R}^n$. Now use your bases to get induced bases of these spaces.

(6) Show that the space $S^2 L^\perp$ also has a proper subspace that is invariant under the action of $\mathfrak{S}_n$ (and the orthogonal complement of this subspace is invariant as well).

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