Final exam, Math 170B - Winter 2015
Instructor: Eviatar B. Procaccia

First name: ______________________________

Last name: ______________________________

Student ID number: ______________________________

Instructions:

- Read problems very carefully. If you have any questions raise your hand.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can (except for Problem 1). This way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.
- You final answers need to be simplified only if this is required in the statement of the problem. Otherwise, you can leave them in any form you wish.
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand (you can write on the back pages).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	0	
Total:	70	

1. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events.

   (a) (3 points) Define $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

   (b) (3 points) Assume that A_n is an increasing sequence and prove that $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$.

   (c) (4 points) Assume that A_n is an increasing sequence and prove that $\mathbf{P}[\lim_{n\to\infty} A_n] = \lim_{n\to\infty} \mathbf{P}[A_n]$.
   Hint: Define $B_i = A_i \setminus A_{i-1}$

2. (a) (5 points) Let F be a strictly increasing CDF of a random variable X. Let $U \sim \mathcal{U}[0,1]$. Define a new random variable $Y(\omega) = F^{-1}(U(\omega))$. What is the distribution of Y?

   (b) (5 points) How would you generate a random variable $Y \sim exp(\lambda)$ given U? Justify your answer.

3. (a) (5 points) Let Y_4 be an Erlang random variable of order 4 with parameter λ. What is the moment generating function (transform) of Y_4?

   (b) (5 points) Let T be an exponential distributed random variable with parameter μ, independent of Y_4. Compute the probability that $Y_4 > T$. Hint: can solve this question without integration.

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequences of random variables with $\mathbf{E}[X_1] = 1/2$ and $\mathrm{Var}[X_i] = 2$.

(a) (5 points) Compute

$$\mathbf{P}\left[\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} > 1\right].$$

(b) (5 points) Does the following limit exist almost surely? If so what is the limit?

$$\lim_{n\to\infty} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{X_1 + X_2 + \cdots + X_n}.$$

5. (10 points) Toss a fair coin repeatedly and stop tossing when three consecutive heads appear. Find the expected number of tosses.

6. 100 people are on a flight. Each person is an adult with probability $3/4$ and a child with probability $1/4$. Adults weight is uniformly distributed on the interval $[120, 280]$ and children's weight is uniformly distributed on the interval $[50, 110]$. Each person's age and weight is independent of everyone else.

   (a) (5 points) Let X_i be the weight of the ith person to board the plane. Compute $\mathbf{E}[X_i]$ and $\mathrm{Var}[X_i]$.

   (b) (5 points) Use Markov's inequality to estimate the probability the total weight is above $30,000$.

7. (a) (3 points) Define almost sure convergence and convergence in probability.
   (b) (3 points) Present an example of a sequence of random variables that converges in probability but not almost surely.
   (c) (4 points) Show that if $X_n \to X$ in probability then there is a subsequence $X_{n_k} \to X$ almost surely.

8. Let $a > c > 0$ and $b > 0$. Show that the number of paths which youch the line $x = a$ and then lead to (n, c) without having touched the line $x = -b$ equals $N_{n,2a-c} - N_{n,2a+2b+c}$. Note that this includes paths touching the line $x = -b$ before the line $x = a$.