You may use your class notes and texts, but not collaboration or internet.

Your score on this supplement will be added to your test 1 grade.

Let

\[ CO(n) := \{ f \in \text{End}(\mathbb{R}^n) \mid \langle f(v), f(w) \rangle = \lambda_f \langle v, w \rangle, \forall v, w \in V, \text{ for some } \lambda_f \neq 0 \} \]

\( CO(n) \) is a Lie group, called the \textit{conformal group}. It is the set of linear maps \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) that preserve angles between vectors.

1. Compute the Lie algebra \( \mathfrak{o}(n) \) as a linear subspace of the space of \( n \times n \) matrices.

2. Show that the Riemannian metric on \( S^n \setminus N \) induced from the inclusion into \( \mathbb{R}^{n+1} \) is preserved under stereographic projection (\( N \) is the north pole), i.e. stereographic projection is a conformal map. Thus the local conformal geometry of surfaces in \( \mathbb{R}^n \) is the same as the local conformal geometry of surfaces in \( S^n \).

3. Let \( \mathbb{R}^{n+2} \) with linear coordinates \( (x^0, \ldots, x^{n+2}) \) be equipped with the following symmetric bilinear form \( (x, y) = -x^0y^0 + x^1y^1 + \cdots + x^{n+1}y^{n+1} \). Show that \( S^n \) is diffeomorphic to \( \mathbb{P}\{x \in \mathbb{R}^{n+2}, x^0 > 0, \text{ and } (x, x) = 0\} \).

4. Show that the identification in the previous problem induces the standard conformal structure on \( S^n \). Thus just as it was natural to study moving frames for \( \mathbb{P}^n \) via \( GL_{n+1}\mathbb{R} \), it is natural to study moving frames for conformal \( S^n \) via \( GL_{n+2}\mathbb{R} \).