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AN OVERVIEW OF MATHEMATICAL ISSUES ARISING IN THE GEOMETRIC COMPLEXITY THEORY APPROACH TO VP $\neq$ VNP*

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Abstract. We discuss the geometry of orbit closures and the asymptotic behavior of Kronecker coefficients in the context of the geometric complexity theory program to prove a variant of Valiant's algebraic analogue of the $\mathbf{P} \neq \mathbf{NP}$ conjecture. We also describe the precise separation of complexity classes that their program proposes to demonstrate.

Key words. geometric complexity theory, **P** vs. **NP**, geometric invariant theory, orbit closure, Kronecker coefficient, determinant, permanent

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1. Introduction.

In a series of papers [45, 46, 47, 48, 49, 50, 51, 52, 53], Mulmuley and Sohoni outline an approach to the **P** vs. **NP** problem that they call the *geometric complexity theory (GCT)* program. The starting point is Valiant's conjecture [66] (see also [20, 8]) that the permanent hypersurface in m^2 variables (i.e., the set of $m \times m$ matrices X with $\mathrm{perm}_m(X) = 0$) cannot be realized as an affine linear section of the determinant hypersurface in $n(m)^2$ variables with $n(m)$ a polynomial function of m. Their program (at least up to [46]) translates the problem of proving Valiant's conjecture to proving a conjecture in representation theory. In this paper we give an exposition of the program outlined in [45, 46], present the representation-theoretic conjecture in detail, and present a framework for reducing their representation theory questions to easier questions by taking more geometric information into account. We also precisely identify the complexity problem the GCT approach proposes to solve and how it compares to Valiant's original conjecture, and discuss related issues in geometry that arise from their program. The goal of this paper is to clarify the state of the art and to identify steps that would further advance the program using recent advances in geometry and representation theory.

The GCT program translates the study of the hypersurfaces

$$\{\mathrm{perm}_m = 0\} \subset \mathbb{C}^{m^2} \quad \text{and} \quad \{\det_n = 0\} \subset \mathbb{C}^{n^2}$$

to a study of the orbit closures

$$\overline{GL_{n^2} \cdot [\ell^{n-m}\mathrm{perm}_m]} \subset \mathbb{P}(S^n\mathbb{C}^{n^2}) \quad \text{and} \quad \overline{GL_{n^2} \cdot [\det_n]} \subset \mathbb{P}(S^n\mathbb{C}^{n^2}),$$

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where $S^n\mathbb{C}^{n^2}$ denotes the space of homogeneous polynomials of degree n in n^2 variables. Here ℓ is a linear coordinate on $\mathbb{C}$, and one takes any linear inclusion $\mathbb{C}\oplus\mathbb{C}^{m^2}\subset\mathbb{C}^{n^2}$ to have $\ell^{n-m}\mathrm{perm}_m$ be a homogeneous degree n polynomial on $\mathbb{C}^{n^2}$. Mulmuley and Sohoni observe that a variant of Valiant's hypothesis would be proved if one could show the following.

CONJECTURE 1 (see [45]). *There does not exist a constant $c \geq 1$ such that for sufficiently large m,*

$$\overline{GL_{m^{2c}}\cdot[\ell^{m^c-m}\mathrm{perm}_m]}\subset\overline{GL_{m^{2c}}\cdot[\det_{m^c}]}.$$

It is known that $\overline{GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]}\subset\overline{GL_{n^2}\cdot[\det_n]}$ for $n=\mathcal{O}(m^2 2^m)$; see Remark 9.3.3.

For a closed subvariety X of $\mathbb{P}V$, let $\hat{X}\subset V$ denote the cone over X. Let $I(\hat{X})\subset Sym(V^*)$ be the ideal of polynomials vanishing on $\hat{X}$, and let $\mathbb{C}[X]=Sym(V^*)/I(\hat{X})$ denote the homogeneous coordinate ring. For two closed subvarieties X,Y of $\mathbb{P}V$, one has $X\subset Y$ iff $\mathbb{C}[Y]$ surjects onto $\mathbb{C}[X]$ by restriction of polynomial functions.

The GCT program sets out to prove the following.

CONJECTURE 2 (see [45]). *For all $c\geq 1$ and for infinitely many m there exists an irreducible $GL_{m^{2c}}$-module appearing in $\mathbb{C}[\overline{GL_{m^{2c}}\cdot[\ell^{m^c-m}\mathrm{perm}_m]}]$ but not appearing in $\mathbb{C}[\overline{GL_{m^{2c}}\cdot[\det_{m^c}]}]$.*

Both varieties occurring in Conjecture 2 are invariant under $GL_{m^{2c}}$, so their coordinate rings are $GL_{m^{2c}}$-modules. Conjecture 1 is a straightforward consequence of Conjecture 2 by Schur's lemma.

A program to prove Conjecture 2 is outlined in [46], which also contains a discussion of why the desired irreducible modules (called *representation theoretic obstructions*) should exist. This is closely related to a separability question [46, Conjecture 12.4] that we will not address in this paper.

There are several paths one could take to try to find such a sequence of modules. The path chosen in [46] is to consider $SL_{n^2}\cdot\det_n$ and $SL_{m^2}\cdot\mathrm{perm}_m$ because on one hand, their coordinate rings can be determined in principle using representation theory, and on the other hand, they are closed affine varieties. Mulmuley and Sohoni observe that any irreducible SL_{n^2}-module appearing in $\mathbb{C}[SL_{n^2}\cdot\det_n]$ must also appear in the degree δ part of the graded SL_{n^2}-module $\mathbb{C}[\overline{GL_{n^2}\cdot[\det_n]}]_\delta$ for some δ. Regarding the permanent, for $n>m$, $SL_{n^2}\cdot\ell^{n-m}\mathrm{perm}_m$ is not closed, so they develop machinery to transport information about $\mathbb{C}[SL_{m^2}\cdot\mathrm{perm}_m]$ to $\mathbb{C}[\overline{GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]}]$; in particular they introduce a notion of *partial stability*.

We make a close study of how one might exploit partial stability to determine the GL_{n^2}-module decomposition of $\mathbb{C}[\overline{GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]}]$ in section 5. We also discuss a more elementary approach to studying which modules in $\mathbb{C}[GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]]$ could appear in the degree δ part of $\mathbb{C}[\overline{GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]}]$. One could get more information from the elementary approach if one could solve the *extension problem* of determining which functions on the orbit $GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]$ extend to its orbit closure. In general the extension problem is very difficult; we discuss it in section 7.

We express the restrictions on modules appearing in $\mathbb{C}[\overline{GL_{n^2}\cdot[\ell^{n-m}\mathrm{perm}_m]}]$ that we do have, as well as our information regarding $\mathbb{C}[\overline{GL_{n^2}\cdot[\det_n]}]$, in terms of *Kronecker coefficients* and *symmetric Kronecker coefficients* that we introduce in section 5.2. Kronecker coefficients are defined as the multiplicities occurring in tensor products of representations of symmetric groups. We review all relevant information regarding these coefficients that we are aware of in section 8. Unfortunately, from this information, we are currently unable to see how one could prove Conjecture 2 in the case $c = 1$ (which is straightforward by other means), let alone for all c. Nevertheless, we have found the GCT program a beautiful source of inspiration for future work.

This program is beginning to gain the attention of the mathematical community, for example in the recent article [56], where an algorithm is given for determining if one orbit is in the closure of another, and in [6], where a conjecture of Mulmuley regarding Kronecker coefficients is disproved and, in an appendix by Mulmuley, a modified conjecture is proposed. Since the original submission of this paper in July of 2009, there have been several developments [35, 11, 12, 33, 13] whose relevance we note where appropriate in the body of the paper.

2. Overview. We begin, in section 3, by establishing notation and reviewing basic facts from representation theory that we use throughout. In section 4 we discuss coordinate rings of orbits and orbit closures, and in section 5 we make a detailed study of the cases at hand. In section 6.1 we state the theorems in [46] and also give an overview of their proofs. The consequences of partial stability can be viewed from the perspective of the *collapsing method* for computing coordinate rings (and syzygies), which we discuss in section 6.2.

While [46] is primarily concerned with $SL_{n^2} \cdot \det_n$ and a corresponding closed orbit related to the permanent, we also study the coordinate rings of the orbits of the general linear group GL_{n^2}. The GL_{n^2}-orbits have the disadvantage of not being closed in general, so one must deal with the *extension problem*, which we discuss in section 7, but they have the advantage of having a graded coordinate ring.

In the studies of the coordinate rings of the permanent and determinant, *Kronecker coefficients* play a central role. We discuss what is known about the relevant Kronecker coefficients in section 8. In section 9 we give a brief outline of the relevant algebraic complexity theory involved here. We explain Valiant's conjecture $\mathbf{VP} \neq \mathbf{VNP}$ and how this precisely relates to the conjecture regarding projecting the determinant to the permanent, and we formulate Conjecture 1 as the separation of complexity classes $\overline{\mathbf{VP}_{\mathrm{ws}}} \neq \mathbf{VNP}$.

3. Notation and preliminaries. Throughout we work over the complex numbers $\mathbb{C}$. Let V be a complex vector space, let $GL(V)$ denote the general linear group of V, let $v \in V$, and let $G \subseteq GL(V)$ be a subgroup. We let $G \cdot v \subset V$ denote the orbit of v, $\overline{G \cdot v} \subset V$ its Zariski closure, and $G(v) \subset G$ the stabilizer of v, so $G \cdot v \simeq G/G(v)$. Write $\mathbb{C}[G \cdot v]$ (resp., $\mathbb{C}[\overline{G \cdot v}]$) for the ring of regular functions on $G \cdot v$ (resp., $\overline{G \cdot v}$). By restriction, there is a surjective map $Sym(V^*) \to \mathbb{C}[\overline{G \cdot v}]$.

It will be convenient to switch back and forth between vector spaces and projective spaces. $\mathbb{P}V$ denotes the space of lines through the origin in V. If $v \in V$ is nonzero, let $[v] \in \mathbb{P}V$ denote the corresponding point in projective space, and if $x \in \mathbb{P}V$, let $\hat{x} \subset V$ denote the corresponding line. A linear action of G on V induces an action of G on $\mathbb{P}V$; let $G([v])$ denote the stabilizer of $[v] \in \mathbb{P}V$. If $Z \subset \mathbb{P}V$ is a subset, let $\hat{Z} \subset V$ denote the corresponding cone in V.

We will be concerned with the space of homogeneous polynomials of degree n in n^2 variables, $V = S^n(\mathrm{Mat}^*_{n\times n}) = S^n W$. Here $\mathrm{Mat}_{n\times n}$ denotes the space of $n \times n$-matrices, $S^n W$ denotes the space of homogeneous polynomials of degree n on W^*, and $G = GL(W)$. Our main points of interest will be $x = [\det_n]$ and $x = [\ell^{n-m}\mathrm{perm}_m]$, where $\det_n \in S^n(\mathrm{Mat}^*_{n\times n})$ is the determinant of an $n \times n$ matrix, $\mathrm{perm}_m \in S^m(\mathrm{Mat}^*_{m\times m})$ is the permanent, we have made a linear inclusion $\mathrm{Mat}_{m\times m} \subset \mathrm{Mat}_{n\times n}$, and ℓ is a linear form on $\mathrm{Mat}_{n\times n}$ annihilating the image of $\mathrm{Mat}_{m\times m}$.

For a reductive group G, the set of dominant integral weights Λ_G^+ indexes the irreducible (finite dimensional) G-modules (see, e.g., [19, 30]), and for $\lambda \in \Lambda_G^+$, $V_\lambda(G)$ denotes the irreducible G-module with highest weight λ; if G is understood, we just write V_λ. If $H \subset G$ is a subgroup and V a G-module, let $V^H := \{v \in V \mid \forall h \in H\ h{\cdot}v = v\}$ denote the space of H-invariant vectors. For a G-module V, let $\mathrm{mult}(V_\lambda(G), V)$ denote the multiplicity of the irreducible representation $V_\lambda(G)$ in V.

The weight lattice Λ_{GL_M} of GL_M is $\mathbb{Z}^M$, and the dominant integral weights $\Lambda^+_{GL_M}$ can be identified with the M-tuples $(\pi_1, \dots, \pi_M)$ with $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_M$. For future reference, we note that

$$V_{(\pi_1,\dots,\pi_M)}(GL_M)^* = V_{(-\pi_M,\dots,-\pi_1)}(GL_M). \tag{3.0.1}$$

The polynomial irreducible representations of GL_M are the Schur modules $S_\pi\mathbb{C}^M$, indexed by partitions $\pi = (\pi_1, \dots, \pi_M)$ with $\pi_1 \geq \pi_2 \geq \dots \geq \pi_M \geq 0$. To get all the rational irreducible representations, we need to twist by negative powers of the determinant. This introduces some redundancies since $S_\pi\mathbb{C}^M \otimes (\det\mathbb{C}^M)^{\otimes k} = S_{\pi+(k,\dots,k)}\mathbb{C}^M$. To avoid them, we consider the modules $S_\pi\mathbb{C}^M \otimes (\det\mathbb{C}^M)^{\otimes k}$ with $k \in \mathbb{Z}$ and $\pi = (\pi_1, \dots, \pi_{M-1}, 0)$. Moreover, we write our partitions as $\pi = (\pi_1, \dots, \pi_N)$ with the convention that $\pi_1 \geq \cdots \geq \pi_N > 0$, and we let $|\pi| = \pi_1 + \cdots + \pi_N$ and $\ell(\pi) = N$. We also write $\pi \vdash_m d$ to express that π is a partition of size $|\pi| = d$ and such that $\ell(\pi) \leq m$. The notation $\pi \mapsto \pi'$ means that $\pi_1 \geq \pi'_1 \geq \pi_2 \geq \pi'_2 \geq \cdots \geq 0$.

The irreducible SL_M-modules are obtained by restricting the irreducible GL_M-modules, but beware that this is insensitive to a twist by the determinant. The weight lattice of Λ_{SL_M} of SL_M is $\mathbb{Z}^{M-1}$, and the dominant integral weights $\Lambda^+_{SL_M}$ are the nonnegative linear combinations of the fundamental weights $\omega_1, \dots, \omega_{M-1}$. A Schur module $S_\pi\mathbb{C}^M$ considered as an SL_M-module has highest weight

$$\lambda = \boldsymbol{\lambda}(\pi) = (\pi_1 - \pi_2)\omega_1 + (\pi_2 - \pi_3)\omega_2 + \cdots + (\pi_{M-1} - \pi_M)\omega_{M-1}.$$

We write $S_\pi\mathbb{C}^M = V_{\boldsymbol{\lambda}(\pi)}(SL_M)$ or simply $V_{\boldsymbol{\lambda}(\pi)}$ if SL_M is clear from the context.

Let $\boldsymbol{\pi}(\lambda)$ denote the smallest partition such that the GL_M-module $S_{\boldsymbol{\pi}(\lambda)}\mathbb{C}^M$, considered as an SL_M-module, is V_λ. That is, $\boldsymbol{\pi}$ is a map from $\Lambda^+_{SL_M}$ to $\Lambda^+_{GL_M}$, mapping $\lambda = \sum_{j=1}^{M-1} \lambda_j\omega_j$ to

$$\boldsymbol{\pi}(\lambda) = \left(\sum_{j=1}^{M-1} \lambda_j, \sum_{j=2}^{M-1} \lambda_j, \dots, \lambda_{M-1} \right).$$

4. Stabilizers and coordinate rings of orbits. As mentioned in the introduction, [46] proposes to study the rings of regular functions on $\overline{GL_{n^2} \cdot \det_n}$ and $\overline{GL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m}$ by first studying the regular functions on the closed orbits $SL_{n^2} \cdot \det_n$ and $SL_{m^2} \cdot \ell^{n-m}\mathrm{perm}_m$. In this section we review facts about the coordinate ring of a homogeneous space and stability of orbits, record observations in [46] comparing closed $SL(W)$-orbits and $GL(W)$-orbit closures, state their definition of partial stability, and record Theorem 4.5.5, which illustrates a potential utility of partial stability.

Throughout this section, unless otherwise specified, G will denote a reductive group and V a G-module.

4.1. Coordinate rings of homogeneous spaces. The coordinate ring of a reductive group G has a left-right decomposition, as a $(G-G)$-bimodule,

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda_G^+} V_\lambda^* \otimes V_\lambda, \tag{4.1.1}$$

where V_λ denotes the irreducible G-module of highest weight λ.

Let $H \subset G$ be a closed subgroup. The coordinate ring of the homogeneous space G/H is obtained by taking (right) H-invariants in (4.1.1) giving rise to the (left) G-module decomposition

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \bigoplus_{\lambda \in \Lambda_G^+} V_\lambda^* \otimes V_\lambda^H = \bigoplus_{\lambda \in \Lambda_G^+} (V_\lambda^*)^{\oplus \dim V_\lambda^H}. \tag{4.1.2}$$

The second equality holds because V_λ^H is a trivial (left) G-module. See [32, Chap. II, sect. 3, Thm. 3] or [57, section 7.3] for an exposition of these facts.

4.2. Orbits with reductive stabilizers. Let G be a reductive group, let V be an irreducible G-module, and let $v \in V$ be such that its stabilizer $G(v)$ is reductive. Then $G \cdot v = G/G(v) \subset V$ is an affine variety [43, Cor., p. 206]. The complement of an affine variety in a complete variety is always of pure codimension one (see [23, Chap. 2, Prop. 3.1]). From this it follows that the boundary of $G \cdot v$ is empty or has pure codimension one in $\overline{G \cdot v}$. Indeed, we can complete V by a hyperplane at infinity and take the closure in the resulting projective space. Then we have to throw away the components at infinity of the boundary, and for the other components we remove their intersection with the hyperplane at infinity. This preserves the pure codimension one property.

4.3. Stability. Following Kempf [28], a nonzero vector $v \in V$ is said to be *G-stable* if the orbit $G \cdot v$ is closed. We then also say that $[v] \in \mathbb{P}V$ is G-stable. If $V = S^d W$ for $\dim W > 3$, $d > 3$, and $v \in V$ is generic, then by [55] its stabilizer in $SL(W)$ is finite, and by [32, Chap. II, sect. 4.3.D, Thm. 6 p. 142], this implies that v is stable with respect to the $SL(W)$-action.

Kempf's criterion [28, Cor. 5.1] states that if G does not contain a nontrivial central one-parameter subgroup, and the stabilizer $G([v])$ is not contained in any proper parabolic subgroup of G, then v is G-stable. We will apply Kempf's criterion to the determinant in section 5.2 and to the permanent in section 5.5.

If v is G-stable, then of course $\mathbb{C}[G \cdot v] = \mathbb{C}[\overline{G \cdot v}]$. The former is an intrinsic object with the above representation-theoretic description, while the latter is the quotient of the space of all polynomials on V by those vanishing on $G \cdot v$.

4.4. *GL(W)*- vs. *SL(W)*-orbits. Let V be a $GL(W)$-module, and let $v \in V$ be nonzero. Suppose that the homotheties in $GL(W)$ act nontrivially on v. Then the orbit $GL(W) \cdot v$ is never stable, as it contains the origin in its closure.

Assume that v is $SL(W)$-stable, so $\mathbb{C}[SL(W) \cdot v] = \mathbb{C}[\overline{SL(W) \cdot v}]$ can be described using (4.1.2). Unfortunately, the ring $\mathbb{C}[SL(W) \cdot v]$ is not graded. However, $\overline{GL(W) \cdot v}$ is a cone over $SL(W) \cdot v$ with vertex the origin. The coordinate ring of $\overline{GL(W) \cdot v}$ is equipped with a grading because $\overline{GL(W) \cdot v}$ is invariant under rescaling, so any polynomial vanishing on it must also have each of its homogeneous components vanishing on it separately. In fact this coordinate ring is the image of a surjective map $Sym(V^*) = \mathbb{C}[V] \to \mathbb{C}[\overline{GL(W) \cdot v}]$, given by restriction of polynomial functions, and this map respects the grading.

Consider the restriction map $\mathbb{C}[\overline{GL(W)\cdot v}]_\delta \to \mathbb{C}[SL(W)\cdot v]$. It is injective for all δ because a homogeneous polynomial vanishing on an affine variety vanishes on the cone over it. On the other hand, because $SL(W)\cdot v$ is a closed subvariety of $\overline{GL(W)\cdot v}$, restriction of functions yields a surjective map $\mathbb{C}[\overline{GL(W)\cdot v}] \to \mathbb{C}[SL(W)\cdot v]$. Both $\mathbb{C}[\overline{GL(W)\cdot v}]_\delta$ and $\mathbb{C}[SL(W)\cdot v]$ are $SL(W)$-modules (as $\overline{GL(W)\cdot v}$ is also an $SL(W)$-variety), and the map between them is an $SL(W)$-module map because the $SL(W)$-action on functions commutes with restriction.

Summing over all δ yields a surjective $SL(W)$-module map

$$\bigoplus_\delta \mathbb{C}[\overline{GL(W)\cdot v}]_\delta \to \mathbb{C}[SL(W)\cdot v]$$

that is injective in each degree δ. We have the following consequence observed in [46].

PROPOSITION 4.4.1. *Let V be a $GL(W)$-module and let $v \in V$ be $SL(W)$-stable. An irreducible $SL(W)$-module appears in $\mathbb{C}[SL(W)\cdot v]$ iff it appears in $\mathbb{C}[\overline{GL(W)\cdot v}]_\delta$ for some δ.*

In contrast to the case of $SL(W)$, if an irreducible module occurring in $\mathbb{C}[GL(W)\cdot v]$ also occurs in $\mathbb{C}[\overline{GL(W)\cdot v}] \subset Sym(V^*)$, we can recover the degree it appears in. Consider the case $V = S^dW$; then a $GL(W)$-module $S_\pi W$ can occur only in $\mathbb{C}[\overline{GL(W)\cdot v}]$ if $|\pi| = \delta d$ for some δ, and in that case it can appear only in $\mathbb{C}[\overline{GL(W)\cdot v}]_\delta$ (see the example in section 5.1 below).

4.5. Partial stability and an application. Let V be a $GL(W)$-module and let $v, w \in V$ be $SL(W)$-stable points. Equation (4.1.2) and Proposition 4.4.1 imply the following observation: $w \notin \overline{GL(W)\cdot v}$ (equivalently $\overline{GL(W)\cdot w} \not\subset \overline{GL(W)\cdot v}$) if there is an $SL(W)$-module that contains an $SL(W)(w)$-invariant that does not contain an $SL(W)(v)$-invariant. As discussed below, $\det_n$ is $SL(W)$-stable, and while $\ell^{n-m}\mathrm{perm}_m$ is not $SL(W)$-stable, it is what is called *partially stable* in [46], which allows one to attempt to search for such modules as we now describe.

DEFINITION 4.5.1 (see [46]). *Let G be a reductive group and let V be a G-module. Let $P = KU$ be a Levi decomposition of a parabolic subgroup P of G. Let R be a reductive subgroup of K. We say that $[v] \in \mathbb{P}V$ is (R, P)-stable if it satisfies the following two conditions:*

1. *$U \subset G([v]) \subset P$.*
2. *v is stable under the restricted action of R; that is, $R\cdot v$ is closed.*

Example 4.5.2. If $x \in S^dW'$ is a generic element and $W' \subsetneq W$ is a linear inclusion, then x is not $SL(W)$-stable, but it is $(SL(W'), P)$ stable for P the parabolic subgroup of $SL(W)$ fixing the subspace $W' \subset W$. This follows from section 4.3, assuming $\dim W' > 3$ and $d > 3$.

Example 4.5.3. Let $W = A \oplus A' \oplus B$, $A = E \otimes F \simeq \mathrm{Mat}_{m\times m}$, $\dim A' = 1$, and $G = GL(W)$. Let $\ell \in A'$ such that $\ell \neq 0$. It follows from section 4.3 that $\ell^{n-m}\mathrm{perm}_m \in S^n\mathrm{Mat}^*_{n\times n}$ is (R, P)-stable for $R = SL(A)$ and P the parabolic subgroup of G preserving $A \oplus A'$, whose Levi factor is $K = (GL(A \oplus A') \times GL(B))$.

The point of partial stability is that, since the point v is assumed to be R-stable, the problem of determining the multiplicities of the irreducible modules $V_\nu(R)$ in $\mathbb{C}[\overline{R\cdot v}]$ is reduced to the problem of determining the dimension of $V_\nu(R)^{R(v)}$. In the case $R = K$, these are also the multiplicities of the corresponding irreducible representations in the coordinate ring $\mathbb{C}[\overline{G\cdot v}]$.

We will now state a central result of [46] (Theorem 6.1.5 below) in the special case that will be applied to $\ell^{n-m}\mathrm{perm}_m$. We first need to recall the classical Pieri formula (see, e.g., [67, Proposition 2.3.1] for a proof).

PROPOSITION 4.5.4. *For* $\dim A' = 1$, *one has the* $GL(A) \times GL(A')$*-module decomposition*

$$S_\pi(A \oplus A') = \bigoplus_{\pi \mapsto \pi'} S_{\pi'} A \otimes S^{|\pi|-|\pi'|} A',$$

where the notation $\pi \mapsto \pi'$ *means that* $\pi_1 \geq \pi'_1 \geq \pi_2 \geq \pi'_2 \geq \cdots \geq 0$.

THEOREM 4.5.5. *Let* $W = A \oplus A' \oplus B$, $\dim A = \mathbf{a}$, $\dim A' = 1$, $z \in S^{d-s}A$, *and* $\ell \in A' \setminus \{0\}$. *Assume* z *is* $SL(A)$*-stable. Write* $v = \ell^s z$. *Set* $R = SL(A)$, *and take* P *to be the parabolic of* $GL(W)$ *preserving* $A \oplus A'$, *so* $K = GL(A \oplus A') \times GL(B)$, *and* z *is* (R, P)*-stable.*

1. *A module* $S_\nu W^*$ *occurs in* $\mathbb{C}[\overline{GL(W) \cdot v}]_\delta$ *iff* $S_\nu(A \oplus A')^*$ *occurs in* $\mathbb{C}[\overline{GL(A \oplus A') \cdot v}]_\delta$. *There is then a partition* ν' *such that* $\nu \mapsto \nu'$ *and* $V_{\boldsymbol{\lambda}(\nu')}(SL(A)) \subset \mathbb{C}[\overline{SL(A) \cdot [v]}]_\delta$.
2. *Conversely, if* $V_\lambda(SL(A)) \subset \mathbb{C}[\overline{SL(A) \cdot [v]}]_\delta$, *then there exist partitions* π, π' *such that* $S_\pi W^* \subset \mathbb{C}[\overline{GL(W) \cdot [v]}]_\delta$, $\pi \mapsto \pi'$, *and* $\boldsymbol{\lambda}(\pi') = \lambda$.
3. *A module* $V_\lambda(SL(A))$ *occurs in* $\mathbb{C}[\overline{SL(A) \cdot [v]}]$ *iff it occurs in* $\mathbb{C}[SL(A) \cdot v]$.

This is a special case of Theorem 6.1.4. It establishes a connection between $\mathbb{C}[\overline{GL(W) \cdot v}]$, which we are primarily interested in but we cannot compute, and $\mathbb{C}[SL(A) \cdot v]$, which in principle can be described using (4.1.2).

We will specialize Theorem 4.5.5 to the case $z = \mathrm{perm}_m$ and study the precise conditions to have an $SL(A)$-module in $\mathbb{C}[SL(A) \cdot \mathrm{perm}_m]$ and the corresponding $GL(W)$-modules in $\mathbb{C}[\overline{GL(W) \cdot [\ell^{n-m}\mathrm{perm}_m]}]$. These conditions are expressed in terms of certain special Kronecker coefficients, and we discuss those Kronecker coefficients in section 8.

5. Examples. We study several examples of orbit closures in spaces of polynomials leading up to the cases of interest, namely $\overline{GL_{n^2} \cdot \det_n}$, $\overline{GL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m}$, $SL_{n^2} \cdot \det_n$, and $SL_{m^2} \cdot \ell^{n-m}\mathrm{perm}_m$. We also study the coordinate rings of the orbits $GL_{n^2} \cdot \det_n$ and $GL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m$. For these to be useful, one must deal with an extension problem, but the advantage is that their coordinate rings come equipped with a grading which, when one passes to the closure, indexes the degree.

5.1. Example. Let $W = \mathbb{C}^n$ and let $x \in S^dW$ be generic. We describe the module structure of $\mathbb{C}[GL(W) \cdot x]$ and $\mathbb{C}[SL(W) \cdot x]$ using (4.1.2). If $x \in S^dW$ is generic and $d, n > 3$, then $GL(W)(x) = \{\lambda \mathrm{Id} : \lambda^d = 1\} \simeq \mathbb{Z}_d$, and hence $GL(W) \cdot x \simeq GL(W)/\mathbb{Z}_d$, where $\mathbb{Z}_d$ acts as multiplication by the dth roots of unity; see [55]. (Note that if $x \in S^dW$ is any element, $\mathbb{Z}_d \subset GL(W)(x)$, and thus the calculation here will be useful for other cases.)

We determine the $\mathbb{Z}_d$-invariants in $GL(W)$-modules. Since $S_\pi W$ is a submodule of $W^{\otimes|\pi|}$, $\omega \in \mathbb{Z}_d$ acts on $S_\pi W \otimes (\det W)^{-s}$ by the scalar $\omega^{|\pi|-ns}$. By (4.1.2), we conclude the following equality of $GL(W)$-modules:

$$\mathbb{C}[GL(W) \cdot x] = \bigoplus_{(\pi,s) \ | \ d||\pi|-ns} (S_\pi W^*)^{\oplus \dim S_\pi W} \otimes (\det W^*)^{-s}.$$

Note that $S^\delta(S^dW^*)$ does not contain any negative powers of the determinant, so when we pass to $\mathbb{C}[\overline{GL(W) \cdot x}] = \oplus_\delta S^\delta(S^dW^*)/I_\delta(\overline{GL(W) \cdot x})$ we must lose all terms with $s > 0$; i.e., we have the inclusion of $GL(W)$-modules

$$\mathbb{C}[\overline{GL(W) \cdot x}] \subseteq \bigoplus_{\pi \ | \ d||\pi|} (S_\pi W^*)^{\oplus \dim S_\pi W}.$$

In general there are far fewer modules and multiplicities in $S^\delta(S^dW)$ than on the right-hand side of the same degree, which illustrates the limitation of this information. The above inclusion respects degree in the graded module $\mathbb{C}[\overline{GL(W)\cdot x}]$:

$$\mathbb{C}[\overline{GL(W)\cdot x}]_\delta \subseteq \bigoplus_{\pi \ \mid\ |\pi|=\delta d} (S_\pi W^*)^{\oplus \dim S_\pi W}. \tag{5.1.1}$$

This property still holds for any $x \in S^dW$, proving the assertion in the last paragraph of section 4.4.

Regarding $SL(W)$, note that $SL(W)(x) = GL(W)(x) \cap SL(W) = \mathbb{Z}_c$, where $c = gcd(d,n)$. Thus (4.1.2) implies that

$$\mathbb{C}[\overline{SL(W)\cdot x}] = \mathbb{C}[SL(W)\cdot x] = \bigoplus_{\lambda\in\Lambda^+_{SL(W)}\ \mid\ c||\boldsymbol{\pi}(\lambda)|} (V_\lambda^*)^{\oplus \dim V_\lambda}. \tag{5.1.2}$$

5.2. First main example: $\boldsymbol{GL(W)\cdot \det_n \subset S^nW}$. Write $W = E\otimes F$, with $E = F = \mathbb{C}^n$. The subgroup $H_0 := \{g\otimes h \mid g \in SL(E), h \in SL(F)\}$ of $GL(E\otimes F)$ is obtained as the image of $SL(E)\times SL(F)$ under the morphism $(g,h)\mapsto g\otimes h$. The kernel of this morphism equals $\{(\varepsilon I, \varepsilon^{-1}I) \mid \varepsilon^n = 1\}$, which is isomorphic to the group μ_n of nth roots of unity, so that $H_0 \simeq (SL(E)\times SL(F))/\mu_n$.

Consider the involution $\tau \in GL(E\otimes F)$ defined by $\tau(e\otimes f) = f\otimes e$ (this makes sense since $E = F$). We note that $\tau(g\otimes h)\tau = h\otimes g$, so τ acts nontrivially on H_0 by conjugation. Hence the group $H := H_0\langle\tau\rangle \simeq H_0 \rtimes \mathbb{Z}_2$ is a nontrivial semidirect product.

Frobenius [18] showed that the stabilizer of $\det_n$ in $GL(W)$ equals the group H:

$$GL(W)(\det_n) = H \simeq \big(SL(E)\times SL(F)\big)/\mu_n \rtimes \mathbb{Z}_2. \tag{5.2.1}$$

(See [22] for indications of modern proofs.) We note that if we interpret W as the space of $n\times n$ matrices M, then the first factor acts as $M\mapsto gMh^t$, with $g\in SL(E)$, $h \in SL(F)$, and τ acts by transposition $M\mapsto M^t$.

As observed in [45, Thm. 4.1], $H = GL(W)(\det_n)$ is not contained in any proper parabolic subgroup, so $[\det_n]$ is $SL(W)$-stable by Kempf's criterion; see section 4.3.

Our next goal is to analyze the space $S_\pi(E\otimes F)^H$ of H-invariants. For this, we note that the Schur module $S_\mu E$ associated with a partition $\mu \vdash d$ can be characterized as $S_\mu E = \mathrm{Hom}_{\mathfrak{S}_d}([\mu], E^{\otimes d})$, where $[\mu]$ denotes the irreducible representation of the symmetric group $\mathfrak{S}_d$ associated with μ. Consider the vector space $K^\pi_{\mu\nu} := \mathrm{Hom}_{\mathfrak{S}_d}\big([\pi], [\mu]\otimes[\nu]\big)$ defined for partitions $\mu,\nu,\pi \vdash d$. Its dimension $k_{\pi\mu\nu} := \dim \mathrm{Hom}_{\mathfrak{S}_d}([\pi],[\mu]\otimes[\nu])$ is called the *Kronecker coefficient* associated with the partitions π,μ,ν. Clearly, $k_{\pi\mu\nu}$ equals the multiplicity of $[\pi]$ in the tensor product $[\mu]\otimes[\nu]$ of representations of $\mathfrak{S}_d$. We refer the reader to section 8, and in particular section 8.3, for remarks on special Kronecker coefficients.

The canonical linear map

$$S_\mu E\otimes S_\nu F\otimes K^\pi_{\mu\nu} \to S_\pi(E\otimes F), \alpha\otimes\beta\otimes\gamma \mapsto (\alpha\otimes\beta)\circ\gamma$$

is $GL(E)\times GL(F)$-equivariant (with the trivial action of this group on $K^\pi_{\mu\nu}$). Schur–Weyl duality [19] tells us that the induced canonical map

$$\bigoplus_{\mu,\nu\vdash_m d} S_\mu E\otimes S_\nu F\otimes K^\pi_{\mu\nu} \ \to\ S_\pi(E\otimes F) \tag{5.2.2}$$

is an isomorphism. Briefly, the splitting of the Schur module $S_\pi(E\otimes F)$ with respect to the morphism $GL(E)\times GL(F)\to GL(E\otimes F), (g,h)\mapsto g\otimes h$ is given by

$$S_\pi(E\otimes F) = \oplus_{\mu,\nu}(S_\mu E\otimes S_\nu F)^{\oplus k_{\pi\mu\nu}}. \tag{5.2.3}$$

The action of $\tau\in GL(E\otimes E)$ determines an involution of $S_\pi(E\otimes E)$ (recall $E = F$). We need to understand the corresponding action on the left-hand side of (5.2.2). For this, we note that the isomorphism $[\mu]\otimes[\nu]\to[\nu]\otimes[\mu]$ resulting from exchanging the factors defines a linear map $\sigma^\pi_{\mu\nu}\colon K^\pi_{\mu\nu}\to K^\pi_{\nu\mu}$ such that $\sigma^\pi_{\nu\mu}\sigma^\pi_{\mu\nu} = \mathrm{id}$. It is straightforward to verify that

$$\tau\cdot((\alpha\otimes\beta)\circ\gamma) = (\beta\otimes\alpha)\circ\sigma^\pi_{\mu\nu}(\gamma) \tag{5.2.4}$$

for $\alpha\in S_\mu E$, $\beta\in S_\nu E$, and $\gamma\in K^\pi_{\mu\nu}$. In the case $\mu=\nu$, we get a linear involution $\sigma^\pi_{\mu\mu}$ of $K^\pi_{\mu\mu}$. The subspace of invariants in $K^\pi_{\mu\mu}$ under this involution can be identified with $\mathrm{Hom}_{\mathfrak{S}_d}([\pi],\mathrm{Sym}^2[\mu])$. We define the corresponding *symmetric Kronecker coefficient* as

$$sk^\pi_{\mu\mu} := \dim\mathrm{Hom}_{\mathfrak{S}_d}([\pi],\mathrm{Sym}^2[\mu]). \tag{5.2.5}$$

So $sk^\pi_{\mu\mu}$ equals the multiplicity of $[\pi]$ in the symmetric square $\mathrm{Sym}^2[\mu]$. Note that $sk^\pi_{\mu\mu}\le k_{\pi\mu\mu}$ and the inequality may be strict. We refer the reader to [37] for some examples.

The symmetric Kronecker coefficients for rectangular partitions $\delta^n = (\delta,\dots,\delta)$ (δ appears n times) show up in the description of the irreducible representations occurring in the coordinate ring of the $GL(W)$-orbit of the determinant.

Proposition 5.2.1.

$$\mathbb{C}[GL(W)\cdot\det_n]_{\mathrm{poly}} = \bigoplus_{\delta\ge 0}\ \bigoplus_{\pi\,|\,|\pi|=n\delta}(S_\pi W^*)^{\oplus sk^\pi_{\delta^n\delta^n}}, \tag{5.2.6}$$

$$\mathbb{C}[\overline{GL(W)\cdot\det_n}]_\delta \subseteq \bigoplus_{\pi\,|\,|\pi|=n\delta}(S_\pi W^*)^{\oplus sk^\pi_{\delta^n\delta^n}}, \tag{5.2.7}$$

$$\mathbb{C}[SL(W)\cdot\det_n] = \mathbb{C}[\overline{SL(W)\cdot\det_n}] = \bigoplus_{\lambda\in\Lambda^+_{SL(W)}}(V^*_\lambda)^{\oplus sk^\pi_{\delta^n\delta^n}}, \tag{5.2.8}$$

where in the sum on the third line $\pi = \boldsymbol{\pi}(\lambda)$, $\delta = |\boldsymbol{\pi}(\lambda)|/n$, *and the subscript* poly *refers to the subring of restrictions of polynomial functions.*

Proof. The multiplicity of $S_\pi W^*$ in $\mathbb{C}[GL(W)\cdot\det_n]$ equals $\dim S_\pi(W)^H$ by (4.1.2). Suppose that $|\pi| = \delta n$ for some δ. Equation (5.2.2) implies that

$$(S_\pi(E\otimes F))^{H_0} = (S_\pi(E\otimes F))^{SL(E)\times SL(F)} = S_{\delta^n}E\otimes S_{\delta^n}F\otimes K^\pi_{\delta^n,\delta^n}\simeq K^\pi_{\delta^n,\delta^n}.$$

For this we used that $S_\mu(E)^{SL(E)} = 0$ unless $\mu = (\delta^n)$, in which case $S_\mu(E)^{SL(E)} = \mathbb{C}$. By (5.2.4) the action of the involution τ corresponds to the action of $\sigma^\pi_{\delta^n\delta^n}$ on $K^\pi_{\delta^n,\delta^n}$. Therefore, $\dim(S_\pi(E\otimes F))^H = sk^\pi_{\delta^n\delta^n}$ by the definition of symmetric Kronecker coefficients. Moreover, if n does not divide $|\pi|$, then $(S_\pi(E\otimes F))^{H_0} = 0$. This completes the proof of (5.2.6).

Equation (5.2.7) is now immediate as $\mathbb{C}[\overline{GL(W)\cdot\det_n}]_\delta\subseteq\mathbb{C}[GL(W)\cdot\det_n]_\delta$. Equation (5.2.8) follows from the proof of (5.2.6). ∎

5.3. Example. Suppose $W = A \oplus B$, with $x \in S^d A$ generic. Here and below let $\mathbf{a} = \dim A$ and $\mathbf{b} = \dim B > 0$. Assume $d, \mathbf{a} > 3$. The stabilizer $GL(W)(x)$ of x in $GL(W)$ is of the form

$$GL(W)(x) = \left\{ \begin{pmatrix} \omega \mathrm{Id} & * \\ 0 & * \end{pmatrix} \mid \omega^d = 1 \right\},$$

where the upper $*$ is an arbitrary $\mathbf{a} \times \mathbf{b}$ matrix, and the lower $*$ is an arbitrary $\mathbf{b} \times \mathbf{b}$ invertible matrix. Since there is no control over the lower right-hand block matrix in $GL(W)(x)$, an irreducible $GL(W)$-module $S_\pi W \otimes (\det W)^{\otimes k}$ can contain nontrivial invariants only if $k = 0$, and then these invariants must be contained in $S_\pi A \subset S_\pi W$. Since $GL(W)(x)$ acts on $S_\pi A$ by homotheties, we conclude that

$$\mathbb{C}[GL(W) \cdot x] = \bigoplus_{\pi \ \mid \ d||\pi|, \ \ell(\pi) \leq \mathbf{a}} (S_\pi W^*)^{\oplus \dim S_\pi A}.$$

In particular, all modules $S_\pi W^*$ with $d||\pi|$ and $\ell(\pi) \leq \mathbf{a}$ do occur. The elimination of modules with more than $\mathbf{a}$ parts is due to our variety being contained in a *subspace variety* (defined in section 6.3 below), consistent with Proposition 6.3.2.

For comparison with what follows, we record the following immediate consequence for all δ:

$$\mathbb{C}[\overline{GL(W) \cdot x}]_\delta \subseteq \bigoplus_{\pi \ \mid \ |\pi| = d\delta, \ \ell(\pi) \leq \mathbf{a}} (S_\pi W^*)^{\oplus \dim S_\pi A}. \tag{5.3.1}$$

Since x is not $SL(W)$-stable, we instead use the $(SL(A), P_{\mathbf{a}})$-partial stability of x to obtain further information. Namely, we take $R = SL(A)$, $K = GL(A) \times GL(B)$, and $P_{\mathbf{a}}$ the parabolic preserving A. From (5.1.2) we have a description of $\mathbb{C}[SL(A) \cdot x]$ in terms of $c = gcd(d, \mathbf{a})$. By Theorem 4.5.5, for each dominant integral weight λ of $SL(A)$ such that c divides $|\boldsymbol{\pi}(\lambda)|$, some π with $\boldsymbol{\lambda}(\pi) = \lambda$ must occur in $\mathbb{C}[\overline{GL(W) \cdot x}]$, and by (5.1.1) it occurs in $\mathbb{C}[\overline{GL(W) \cdot x}]_{|\pi|/d}$.

5.4. Example. Suppose $W = A \oplus A' \oplus B$ and $x = z\ell^s \in S^d W$, where $z \in S^{d-s} A$ is generic, and $\dim A' = 1$, $\ell \in A' \setminus \{0\}$. Assume $d - s, \mathbf{a} > 3$. It is straightforward to show that, with respect to bases adapted to the splitting $W = A \oplus A' \oplus B$,

$$GL(W)(x) = \left\{ \begin{pmatrix} \psi \mathrm{Id} & 0 & * \\ 0 & \eta & * \\ 0 & 0 & * \end{pmatrix} \mid \ \eta^s \psi^{d-s} = 1 \right\}.$$

Working as above, we first observe that the $GL(W)(x)$-invariants in $S_\pi W$ must be contained in $S_\pi(A \oplus A')$. By the Pieri formula (Proposition 4.5.4), this is the sum of the $S_{\pi'} A \otimes S^{|\pi| - |\pi'|} A'$ for $\pi \mapsto \pi'$. The action of $GL(W)(x)$ on such a factor is by multiplication with $\psi^{|\pi'|} \eta^{|\pi| - |\pi'|}$; hence we have the conditions for invariance that $|\pi'| = \delta(d - s)$ and $|\pi| = \delta d$ for some δ. We conclude that

$$\mathbb{C}[GL(W) \cdot x] = \bigoplus_{\delta \geq 0} \bigoplus_{\substack{|\pi| = \delta d, \ |\pi'| = \delta(d-s), \\ \pi \mapsto \pi'}} (S_\pi W^*)^{\oplus \dim S_{\pi'} A},$$

$$\mathbb{C}[\overline{GL(W) \cdot x}]_\delta \subseteq \bigoplus_{\substack{|\pi| = \delta d, \ |\pi'| = \delta(d-s), \\ \pi \mapsto \pi'}} (S_\pi W^*)^{\oplus \dim S_{\pi'} A}. \tag{5.4.1}$$

The point x is not $SL(W)$-stable, but is $SL(A)$-stable, and thus is (R,P)-stable for $(R,P) = (SL(A), P_{\mathbf{a}+1})$. Theorem 4.5.5 applied to this case says that if $S_\pi W^* \subset \mathbb{C}[\overline{GL(W)\cdot x}]_\delta$, then we have $S_\pi(A\oplus A')^* \subset \mathbb{C}[\overline{GL(A\oplus A')\cdot x}]_\delta$ and there exists π' such that $\pi \mapsto \pi'$ and $V_{\lambda(\pi')}(SL(A)) \subset \mathbb{C}[SL(A)\cdot x]$. Moreover, by (5.1.2) the latter condition is equivalent to the condition that $c = gcd(d-s,\mathbf{a})$ divides $|\pi'|$.

5.5. Example. Suppose $W = \mathrm{Mat}_{m\times m}$ and $x = \mathrm{perm}_m$. We write $W = E\otimes F$, with $E = F = \mathbb{C}^m$. Let T_E denote the maximal torus of diagonal matrices in $SL(E)$. Its normalizer N_E is the semidirect product of T_E and the Weyl group $\mathcal{W}_E$ of permutation matrices in $GL(E)$. Similarly, let T_F denote the maximal torus of $SL(F)$ and $N_F = T_F \rtimes \mathcal{W}_F$ its normalizer. If we denote by N_0 the image of $N_E\times N_F$ under $GL(E)\times GL(F) \to GL(E\otimes F), (g,h)\mapsto g\otimes h$, then $N_0 \simeq \big(N_E\times N_F\big)/\mu_m$, where μ_m denotes the group of mth roots of unity. Recall from section 5.2 the involution $\tau \in GL(E\otimes F)$, and consider the subgroup $N := N_0\langle\tau\rangle \simeq N_0 \rtimes \mathbb{Z}_2$.

By [42], for $m > 2$, the stabilizer of $\mathrm{perm}_m \in S^m(E\otimes F)$ equals

$$GL(W)(\mathrm{perm}_m) = N \simeq \big(N_E\times N_F\big)/\mu_m \rtimes \mathbb{Z}_2. \tag{5.5.1}$$

(It is stated in [45] that the stabilizer is found in [44], although this is not correct. A shorter proof of (5.5.1) is given in [4].)

In [45, Thm. 4.7] it is observed that $SL(W)(\mathrm{perm}_m)$ is not contained in any proper parabolic subgroup of $SL(W)$, so perm_m is $SL(W)$-stable by Kempf's criterion; see section 4.3.

Consider the Schur module $S_\mu E$ corresponding to a partition $\mu \vdash_m \delta m$. Then the zero weight space $(S_\mu E)_0 := (S_\mu E)^{T_E}$ of $S_\mu E$ with respect to the $SL(E)$-action is nonzero. The group $\mathcal{W}_E$ acts on $(S_\mu E)_0$, and we shall denote by $p_\mu := \dim(S_\mu E)_0^{\mathcal{W}_E}$ the dimension of the space of its $\mathcal{W}_E$-invariants. In fact, Corollary 8.4.2, stated later on, identifies p_μ as the following *plethysm coefficient*:

$$p_\mu = \mathrm{mult}(S_\mu E, S^m(S^\delta E)).$$

DEFINITION 5.5.1. *Define* $\Sigma_{\mathrm{perm}_m} \subset \Lambda^+_{GL_{m^2}}$ *to be the set of partitions* π *such that the following hold:*

1. $|\pi| = \delta m$ *for some* $\delta\in\mathbb{N}$*;*
2. *there exist* $\mu,\nu \vdash_m \delta m$ *with* $p_\mu p_\nu \neq 0$ *and either*
   (i) $k_{\pi\mu\nu}\neq 0$ *if* $\mu\neq\nu$ *or*
   (ii) $sk^\pi_{\mu\mu}\neq 0$ *if* $\mu=\nu$.

For $\pi\in\Sigma_{\mathrm{perm}_m}$*, define*

$$\mathrm{mult}_\pi = \frac{1}{2}\sum_{\mu\neq\nu} k_{\pi\mu\nu}p_\mu p_\nu + \sum_\mu sk^\pi_{\mu\mu}\binom{p_\mu+1}{2}.$$

Note that $\mathrm{mult}_\pi \geq 1$ *for* $\pi\in\Sigma_{\mathrm{perm}_m}$*. Finally let* $\Sigma^S_{\mathrm{perm}_m} = \boldsymbol{\pi}^{-1}(\Sigma_{\mathrm{perm}_m}) \subset \Lambda^+_{SL_{m^2}}$.

PROPOSITION 5.5.2.

$$\mathbb{C}[GL(W)\cdot\mathrm{perm}_m]_{\mathrm{poly}} = \bigoplus_{\pi\in\Sigma_{\mathrm{perm}_m}} (S_\pi W^*)^{\oplus\,\mathrm{mult}_\pi}, \tag{5.5.2}$$

$$\mathbb{C}[\overline{GL(W)\cdot\mathrm{perm}_m}]_\delta \subseteq \bigoplus_{\substack{\pi\in\Sigma_{\mathrm{perm}_m},\\ |\pi|=\delta m}} (S_\pi W^*)^{\oplus\, mult_\pi}, \tag{5.5.3}$$

$$\mathbb{C}[\overline{SL(W)\cdot\mathrm{perm}_m}] = \mathbb{C}[SL(W)\cdot\mathrm{perm}_m] = \bigoplus_{\lambda\in\Sigma^S_{\mathrm{perm}_m}} (V^*_\lambda)^{\oplus\,\mathrm{mult}_{\boldsymbol{\pi}(\lambda)}}. \tag{5.5.4}$$

Proof. By (4.1.2) we need to show that $\dim S_\pi(W)^{GL(W)(\mathrm{perm}_m)} = mult_\pi$. From (5.2.2) we obtain, using $(S_\mu E)^{T_E} = (S_\mu E)_0$, that

$$(S_\pi(E\otimes F))^{T_E\times T_F} = \bigoplus_{\mu,\nu}(S_\mu E)_0 \otimes (S_\nu F)_0 \otimes K^\pi_{\mu\nu},$$

which implies, using $N_E = T_E \ltimes \mathcal{W}_E$, that

$$(S_\pi(E\otimes F))^{N_E\times N_F} = \bigoplus_{\mu,\nu}(S_\mu E)_0^{\mathcal{W}_E} \otimes (S_\nu F)_0^{\mathcal{W}_F} \otimes K^\pi_{\mu\nu}.$$

For proving (5.5.2), it remains to show that mult_π equals the dimension of the space of τ-invariants of $(S_\pi(E\otimes F))^{N_E\times N_F}$. Put $X_\mu := (S_\mu E)_0^{\mathcal{W}_E}$ to simplify notation. Equation (5.2.4) implies that for $\mu \neq \nu$, the space of τ-invariants

$$\Big(X_\mu\otimes X_\nu\otimes K^\pi_{\mu\nu} \;\oplus\; X_\nu\otimes X_\mu\otimes K^\pi_{\nu\mu}\Big)^\tau$$

projects bijectively onto $X_\mu\otimes X_\nu\otimes K^\pi_{\mu\nu}$. Moreover,

$$\Big(X_\mu\otimes X_\mu\otimes K^\pi_{\mu\mu}\Big)^\tau = \mathrm{Sym}^2(X_\mu)\otimes (K^\pi_{\mu\mu})^\tau.$$

Taking into account $p_\mu = \dim(S_\mu E)_0^{\mathcal{W}_E}$, it follows that $(S_\pi(E\otimes F))^{N_0\langle\tau\rangle} = \mathrm{mult}_\pi$ as claimed in (5.5.2).

Equation (5.5.3) is now immediate as $\mathbb{C}[\overline{GL(W)\cdot \mathrm{perm}_m}]_\delta \subseteq \mathbb{C}[GL(W)\cdot\mathrm{perm}_m]_\delta$. Equation (5.5.4) follows from the proof of (5.5.2). ∎

5.6. Second main example. Let $W = A\oplus A'\oplus B$, $A = E\otimes F \simeq \mathrm{Mat}_{m\times m}$, $\dim A' = 1$, $\dim W = n^2$, and $x = \ell^{n-m}\mathrm{perm}_m$, $\ell \in A'$. With respect to bases adapted to the splitting $W = A\oplus A'\oplus B$,

$$GL(W)(x) = \left\{ \begin{pmatrix} \xi\, GL(W)(\mathrm{perm}_m) & 0 & * \\ 0 & \eta & * \\ 0 & 0 & * \end{pmatrix} \mid \eta^{n-m}\xi^m = 1 \right\}. \tag{5.6.1}$$

Definition 5.6.1. *For $n > m$, define $\Sigma^n_{\mathrm{perm}_m} \subset \Lambda^+_{GL_{n^2}}$ to be the set of partitions π such that the following hold:*

1. *$|\pi| = \delta n$ some $\delta\in\mathbb{N}$;*
2. *there exists $\pi'\in\Sigma_{\mathrm{perm}_m}$ such that $|\pi'| = \delta m$ and $\pi\mapsto\pi'$.*

Moreover, for $\pi\in\Sigma^n_{\mathrm{perm}_m}$ we set

$$\mathrm{mult}^n_\pi = \sum_{\substack{\pi'\in\Sigma_{\mathrm{perm}_m},\ \pi\mapsto\pi' \\ n|\pi'| = m|\pi|}} \mathrm{mult}_{\pi'}.$$

Proposition 5.5.2 and the example in section 5.4 show the following result.

PROPOSITION 5.6.2.

$$\mathbb{C}[GL(W)\cdot \ell^{n-m}\mathrm{perm}_m] = \bigoplus_{\pi\in\Sigma^n_{\mathrm{perm}_m}} (S_\pi W^*)^{\oplus\,\mathrm{mult}^n_\pi},$$

$$\mathbb{C}[\overline{GL(W)\cdot \ell^{n-m}\mathrm{perm}_m}]_\delta \subseteq \bigoplus_{\substack{\pi\in\Sigma^n_{\mathrm{perm}_m},\\ |\pi|=n\delta}} (S_\pi W^*)^{\oplus\,\mathrm{mult}^n_\pi}. \tag{5.6.2}$$

Since $SL(W)\cdot\ell^{n-m}\mathrm{perm}_m$ is not stable, we consider $R = SL(A)$ as in section 5.4. (We could have augmented R by the semisimple part of the stabilizer of $\ell^{n-m}\mathrm{perm}_m$, but this would not yield any new information.) From Theorem 4.5.5 we deduce the following result.

PROPOSITION 5.6.3. *$\ell^{n-m}\mathrm{perm}_m$ is $(SL(A), P_{m^2+1})$-partially stable. Thus for all $\lambda\in\Sigma^n_{\mathrm{perm}_m}$, there exist partitions π,π' such that $\boldsymbol{\lambda}(\pi')=\lambda$, $\pi\mapsto\pi'$, and $S_\pi W^*\subset \mathbb{C}[\overline{GL(W)\cdot\ell^{n-m}\mathrm{perm}_m}]$.*

Since in Proposition 5.6.3 we have no information about in which degree a module appears, for each λ there are an infinite number of π's that could be associated with it. Thus Proposition 5.6.3 may be difficult to utilize in practice.

Proposition 5.6.3 combined with Theorem 4.5.5 gives an explicit description of the Kronecker problem that results from [46] regarding the permanent.

6. Inheritance theorems and desingularizations. In section 6.1 we explain the approach to determine the coordinate ring of an orbit closure outlined in [46]. In section 6.2 we review the geometric method for desingularizing G-varieties by collapsing a homogeneous vector bundle. We then in sections 6.3 and 6.4 give two examples of auxiliary varieties that can be studied with such desingularizations and are useful for the problems at hand. We discuss how this perspective can be used to recover Theorems 6.1.4 and 6.1.5 from [46] and to obtain further information that might be useful.

6.1. Inheritance theorems appearing in [46]. Let $R\subseteq K\subset G$ be as in Definition 4.5.1. We can choose a maximal torus of G in such a way that its intersections with R and K are maximal tori in these subgroups. This allows one to identify weights accordingly, i.e., it induces restriction maps $\Lambda_G\simeq\Lambda_K\to\Lambda_R$, and we impose that $\Lambda_G^+\to\Lambda_K^+\to\Lambda_R^+$.

DEFINITION 6.1.1. *We say that $\nu\in\Lambda^+_G$ lies over $\mu\in\Lambda^+_R$ at v and degree δ if the following hold:*

1. *$V_\mu(R)^*$ and $V_\nu(K)^*$ occur in $\mathbb{C}[\overline{R\cdot[v]}]_\delta$ and $\mathbb{C}[\overline{K\cdot[v]}]_\delta$, respectively;*
2. *$V_\mu(R)^*$ occurs in $V_\nu(K)^*$ considered as an R-module.*

We say that a dominant weight ν of G lies over a dominant weight μ of R at v if this happens for some $\delta>0$.

Example 6.1.2 (Example 4.5.3 cont'd). Let $W = A\oplus A'\oplus B$, dim $A=\mathbf{a}$, dim $A'=1$, $v=\ell^s z$ with $\ell\in A'$, $z\in S^{d-s}A$ such that z is $SL(A)$-stable, so that setting $R=SL(A)$, and P the parabolic subgroup of $GL(W)$ preserving $A\oplus A'$, v is (R,P)-stable. Suppose that a weight in $\Lambda^+_{GL(W)}$ defined by some partition π lies over $\lambda\in\Lambda^+_{SL(A)}$.

First, that $S_\pi W^*$ be contained in $\mathbb{C}[GL(W)\cdot v]$ requires that $\ell(\pi)\le\mathbf{a}+1$ (which will also be justified in section 6.3 by the fact that $\overline{GL(W)\cdot[v]}$ lies in the subspace variety $Sub_{\mathbf{a}+1}(W)$). Second, the condition that $V_\lambda(SL(A))$ be contained in the restriction of $S_\pi(A\oplus A')^*$ requires that $\pi\mapsto\pi'$ for some partition π' such that $\ell(\pi')\leq\mathbf{a}$ and $\lambda(\pi')=\lambda$. Finally, we need $V_\lambda(SL(A))$ to occur in $\mathbb{C}[\overline{SL(A)\cdot[v]}]_\delta$. Theorem 6.1.4 describes when this occurs for some δ.

Definition 6.1.3 (see [46]). *Let $H\subset G$ be a subgroup. We say that a G-module M is H-admissible if it contains a nonzero H-invariant. We let $M^H\subset M$ denote the subspace of H-invariants. Note that an irreducible G-module is H-admissible iff it appears in $\mathbb{C}[G/H]$.*

Theorem 6.1.4 (see [46, Theorem 8.1]). *Let $[v]\in\mathbb{P}V$ be (R,P)-stable. Then the representation $V_\lambda(G)$ occurs in the coordinate ring $\mathbb{C}[\overline{G\cdot[v]}]$ only if λ lies over some $R(v)$-admissible dominant weight μ of R. Conversely, for every $R(v)$-admissible dominant weight μ of R, $\mathbb{C}[\overline{G\cdot[v]}]$ contains $V_\lambda(G)$ for some dominant weight λ of G lying over μ at v.*

Theorem 6.1.4 is a consequence of the following more precise result.

Theorem 6.1.5 (see [46, Theorem 8.2]). *Let $[v]\in\mathbb{P}V$ be (R,P)-stable. Let $P=KU$ be a Levi decomposition of P. Then the following hold:*

1. *A K-module $V_\lambda(K)^*$ occurs in $\mathbb{C}[\overline{K\cdot[v]}]$ only if λ is also dominant for G, and for all δ*

$$\mathrm{mult}(V_\lambda(G)^*,\mathbb{C}[\overline{G\cdot[v]}]_\delta)=\mathrm{mult}(V_\lambda(K)^*,\mathbb{C}[\overline{K\cdot[v]}]_\delta).$$

2. *There are inequalities*

$$\mathrm{mult}(V_\lambda(G)^*,H^0(\overline{G\cdot[v]},\mathcal{O}_{\overline{G\cdot[v]}}(\delta)))\leq\mathrm{mult}(V_\lambda(K)^*,H^0(\overline{K\cdot[v]},\mathcal{O}_{\overline{K\cdot[v]}}(\delta))).$$

3. *A K-module $V_\lambda(K)^*$ can occur in $\mathbb{C}[\overline{K\cdot[v]}]_\delta$ only if $\lambda\in\Lambda_G^+$ lies over some $\mu\in\Lambda_R^+$ at v and degree δ. Conversely, for each R-module $V_\mu(R)^*$ occurring in $\mathbb{C}[\overline{R\cdot[v]}]_\delta$, there exists a G-dominant weight λ lying over μ at v and degree δ.*
4. *An R-module $V_\mu(R)^*$ occurs in $\mathbb{C}[\overline{R\cdot[v]}]$ iff it is $R(v)$-admissible.*

Idea of proof. These statements relate the coordinate rings of the projective orbit closures $\overline{G\cdot[v]}$, $\overline{K\cdot[v]}$, $\overline{R\cdot[v]}$ and of the affine (closed) orbit $R\cdot v$.

In order to prove 1, one observes that the surjective map

$$\mathbb{C}[\overline{G\cdot[v]}]\twoheadrightarrow\mathbb{C}[\overline{K\cdot[v]}]$$

is not only a K-module map but also a P-module map where the P-module structure on the right-hand side is obtained by extending the action of K by the trivial action of U. (This relies on the assumption that $G([v])$ contains U.) Any copy of $V_\lambda(G)^*$ in some $\mathbb{C}[\overline{G\cdot[v]}]_\delta$ maps to a P-module N which is nonzero because if all polynomials in a G-module vanish on $[v]$, they must also vanish on $\overline{G\cdot[v]}$. Dualizing, since the action of U on N is trivial, one gets an injection $N^*\to V_\lambda(G)^U$, whose image is the irreducible module $V_\lambda(K)$. In particular, N^* is irreducible. This implies 1, and its variant 2 is proved in a similar way.

In order to prove 3, one simply observes that the surjection

$$\mathbb{C}[\overline{K\cdot[v]}]\twoheadrightarrow\mathbb{C}[\overline{R\cdot[v]}]$$

is nonzero on any irreducible component of $\mathbb{C}[\overline{K\cdot[v]}]_\delta$ by the same argument as above. So any such $V_\lambda(K)^*$ contributes to $\mathbb{C}[\overline{R\cdot[v]}]_\delta$ by some $V_\mu(R)^*$ for weights μ over which λ lies. Conversely any component of $\mathbb{C}[\overline{R\cdot[v]}]_\delta$ is obtained that way since the restriction map is surjective.

Finally, 4 is a consequence of the fact that $R \cdot v$ is contained in the cone over $\overline{R \cdot [v]}$. Since they are both closed in V, this yields a surjection

$$\mathbb{C}[\overline{R \cdot [v]}] \twoheadrightarrow \mathbb{C}[R \cdot v]$$

and the same argument as for the proof of Proposition 4.4.1 shows that both sides involve the same irreducible modules. □

We emphasize that 4 gives no information of the degree in which a given irreducible module may occur in $\mathbb{C}[\overline{R \cdot [v]}]$.

In this paper we do not discuss 2, whose failure to be an equality is related with the failure of the cone over $\overline{K \cdot [v]}$ to be normal and hence to the type of singularity that occurs at the origin.

There is a connection between the notion of (R, P)-stability and the collapsing method that we discuss in the next subsections. From the latter perspective it is easy to deduce the relationship between $\mathbb{C}[\overline{K \cdot [v]}]$ and $\mathbb{C}[\overline{G \cdot [v]}]$, although the relationship between these and $\mathbb{C}[\overline{R \cdot [v]}]$ is more subtle. It is possible to write alternative proofs of Theorems 6.1.4 and 6.1.5 using the collapsing setup.

The desingularization method could be useful for several reasons. First, it allows one to calculate the multiplicity of an irreducible G-module $V_\lambda(G)$ in each graded component of the coordinate ring of an orbit closure. One could detect that one orbit is not in the closure of the other by comparing these multiplicities. Second, it gives information about the multiplicative structure of the coordinate ring. If an orbit $\mathcal{O}_1$ is in the closure of an orbit $\mathcal{O}_2$, then the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_1]$ is a quotient of $\mathbb{C}[\overline{\mathcal{O}}_2]$ so that every polynomial relation in $\mathbb{C}[\overline{\mathcal{O}}_2]$ still holds in $\mathbb{C}[\overline{\mathcal{O}}_1]$. Finally, desingularization gives information about the singularities of an orbit closure, which are important geometric invariants.

6.2. The collapsing method and its connection with partial stability. The following statement can be extracted from [67, Chap. 5].

THEOREM 6.2.1. *Let $Y \subset \mathbb{P}V$ be a projective variety. Suppose there is a projective variety $\mathcal{B}$ and a vector bundle $q : E \to \mathcal{B}$ that is a subbundle of a trivial bundle $\underline{V} \to \mathcal{B}$ with fiber V, such that the image of the map $\mathbb{P}E \to \mathbb{P}V$ is Y and $\mathbb{P}E \to Y$ is a desingularization of Y. Write $\eta = E^*$ and $\xi = (\underline{V}/E)^*$.*

If the sheaf cohomology groups $H^i(\mathcal{B}, S^\delta\eta)$ are all zero for $i > 0$ and $\delta > 0$, and if the linear maps $H^0(\mathcal{B}, S^\delta\eta) \otimes V^* \to H^0(\mathcal{B}, S^{\delta+1}\eta)$ are surjective for all $\delta \geq 0$, then the following hold:

1. *$\hat{Y}$ is normal, with rational singularities.*
2. *The coordinate ring $\mathbb{C}[\hat{Y}]$ satisfies $\mathbb{C}[\hat{Y}]_\delta \simeq H^0(\mathcal{B}, S^\delta\eta)$.*
3. *If, moreover, Y is a G-variety and the desingularization is G-equivariant, then the identifications above are as G-modules.*

Notations, as above, assume that $v \in V$ is (R, P)-stable. Let $W = \langle K \cdot v\rangle$ be the smallest K-submodule of V containing v. Since v is stabilized by U, and U is normalized by K, W is a P-submodule of V with a trivial U-action. Consider the diagram

$$\begin{array}{ccc} E_W := G \times_P W & \xrightarrow{\ p\ } & G/P \\ \big\downarrow{\scriptstyle q} & & \\ Z_W \subset V, & & \end{array}$$

where E_W is a vector bundle over G/P with fiber W, and $Z_W := q(E_W) = \overline{G{\cdot}W} = G{\cdot}W$. The coordinate ring of Z_W is a subring of $H^0(G/P, Sym(E_W^*))$. In the case when q is a desingularization (i.e., when q is birational), $H^0(G/P, Sym(E_W^*))$ is the normalization of the coordinate ring of Z_W.

The orbit closure $\overline{K \cdot v}$ is a K-stable subset of W, and the method of [67] reduces the calculation of the G-module structure of $\mathbb{C}[\overline{G \cdot v}]$ to the calculation of K-module structure of $\mathbb{C}[\overline{K \cdot v}]$.

6.3. The subspace variety. Let W be a vector space, and for $\mathbf{a} < \dim W$ define

$$Sub_{\mathbf{a}}(S^d W) = \{f \in S^d W \quad | \ \exists\, W' \subset W,\ \dim(W') = \mathbf{a}, f \in S^d W' \subset S^d W\}.$$

$Sub_{\mathbf{a}}(S^d W)$ is a closed subvariety of $S^d W$ which has a natural desingularization given by the total space of a vector bundle over the Grassmannian $Gr(\mathbf{a}, W)$, namely $GL(W) \times_P S^d \mathbb{C}^{\mathbf{a}} = S^d \mathcal{S}$, where $\mathcal{S} \to Gr(\mathbf{a}, W)$ is the tautological subspace bundle over the Grassmannian. In other words, the total space of $S^d\mathcal{S}$ is

$$\{(f, W') \in S^d W \times G(\mathbf{a}, W) \ \mid \ f \in S^d W'\}.$$

Using Theorem 6.2.1 one may determine the generators of the ideal $I(Sub_{\mathbf{a}}(S^d W))$ as follows. For $\phi \in S^d W$ and $\delta < d$, consider the "flattening" $\phi_{\delta, d-\delta} : S^\delta W^* \to S^{d-\delta} W$ via the inclusion $S^d W \subset S^\delta W \otimes S^{d-\delta} W$.

PROPOSITION 6.3.1 (see [67, section 7.2]).

1. *The ideal* $I(Sub_{\mathbf{a}}(S^d W))$ *is the span of all submodules* $S_\pi W^*$ *in* $Sym(S^d W^*)$ *for which* $\ell(\pi) > \mathbf{a}$.
2. $I(Sub_{\mathbf{a}}(S^d W))$ *is generated by* $\Lambda^{\mathbf{a}+1} W^* \otimes \Lambda^{\mathbf{a}+1}(S^{d-1} W^*)$*, which may be considered as the span of the* $(\mathbf{a}+1) \times (\mathbf{a}+1)$ *minors of* $\phi_{1,d-1}$.
3. $Sub_{\mathbf{a}}(S^d W)$ *is normal and Cohen–Macaulay and has rational singularities.*

This proposition implies the following result.

PROPOSITION 6.3.2. *Let* $W' \subset W$ *be a subspace of dimension* $\mathbf{b}$ *and let* $f \in S^d W'$*. Assume that the coordinate ring of the orbit closure* $\overline{GL(W') \cdot f} \subset S^d W'$ *has the* $GL(W')$*-decomposition*

$$\mathbb{C}[\overline{GL(W') \cdot f}] = \bigoplus_{\pi, \ell(\pi) \le \mathbf{b}} (S_\pi W'^*)^{\oplus\, m(\pi)}.$$

Then the coordinate ring of the orbit closure $\overline{GL(W) \cdot f} \subset S^d W$ *has the* $GL(W)$*-decomposition*

$$\mathbb{C}[\overline{GL(W) \cdot f}] = \bigoplus_{\pi, \ell(\pi) \le \mathbf{b}} (S_\pi W^*)^{\oplus\, m(\pi)}.$$

Proof. We actually prove a more precise statement about the two ideals. First note that $\overline{GL(W) \cdot f} \subset Sub_{\mathbf{b}}(S^d W)$, so for all partitions π with $\ell(\pi) > \mathbf{b}$, and $S_\pi W^* \subset Sym(S^d W^*)$, $S_\pi W^* \subset I(\overline{GL(W) \cdot f})$. So henceforth we consider only partitions π with $\ell(\pi) \le \mathbf{b}$.

We will show that $S_\pi W^* \subset I(\overline{GL(W) \cdot f})$ iff $S_\pi W'^* \subset I(\overline{GL(W') \cdot f})$ for any partition π with $\ell(\pi) \le \mathbf{b}$. Assume $|\pi| = d\delta$ (this must be the case for $S_\pi W^*$ to appear in $S^\delta(S^d W^*)$) and $\ell(\pi) \le \mathbf{b}$. Some highest weight vector of $S_\pi W^* \subset S^\delta(S^d W^*)$ lies in $S^\delta(S^d W'^*)$. That it vanishes on $GL(W) \cdot f$ implies it vanishes on $GL(W') \cdot f$ because if we choose a splitting $W = W' \oplus W''$ and write $h \in S^d W$ as $h = h_1 + h_2$ with $h_1 \in S^d W'$, $h_2|_{S^d W'} = 0$, given $p \in S^\delta(S^d W'^*)$, we have $p(h) = p(h_1)$, and

$h \in GL(W) \cdot f$ iff $h_1 \in GL(W') \cdot f$. Finally, an irreducible G-module vanishes on a G-variety iff any highest weight vector vanishes on the variety. $\square$

Remark 6.3.3. The statements above are the special cases of the first part of Theorem 6.1.5 in the case when $W = W' \oplus W''$, and $G = GL(W)$, $K = GL(W') \times GL(W'')$.

Applying Proposition 6.3.2 to $x = \ell^{n-m}\mathrm{perm}_m \in S^n\mathbb{C}^{m^2+1} = W' \subset W = \mathbb{C}^{n^2}$ reduces the problem of determining $\mathbb{C}[\overline{GL(W) \cdot \ell^{n-m}\mathrm{perm}_m}]$ to the one of determining $\mathbb{C}[\overline{GL(W') \cdot \ell^{n-m}\mathrm{perm}_m}]$.

6.4. Polynomials divisible by a linear form. Another ingredient in the collapsing approach to Theorem 6.1.4 is investigating a variety of polynomials divisible by a power of a linear form.

Problem 6.4.1. Let $W' \subset W$ be a subspace of codimension one. Let $\ell \in W \setminus W'$. Let $g \in S^{d-s}W'$. Take $f = \ell^s g \in S^dW$. Compare the decompositions of the coordinate rings of the orbit closures $\overline{GL(W') \cdot g}$ and $\overline{GL(W) \cdot f}$.

A solution to Problem 6.4.1 would allow one to reduce the investigation of the orbit of $\ell^{n-m}\mathrm{perm}_m$ to the orbit closure of the permanent itself.

Consider the subvariety

$$F_s(S^dW) = \{f \in S^dW \mid f = \ell^s g \text{ for some } \ell \in W, g \in S^{d-s}W\}.$$

The variety $F_s(S^dW)$ arises naturally in the GCT program because one is interested in the coordinate ring of $\overline{GL(W) \cdot \ell^{n-m}perm_m}$ which is contained in $F_{n-m}(S^nW)$. The description of the normalization of $F_s(S^dW)$ should be useful because the coordinate ring of $F_s(S^dW)$ is a subring in the coordinate ring of its normalization. This normalization is best understood via a collapsing as follows.

The closed subvariety $F_s(S^dW)$ has a desingularization of the form in Theorem 6.2.1 with $G/P = \mathbb{P}W$, i.e., P is the parabolic subgroup of GL_n stabilizing a subspace of dimension one, and the bundle $\eta = S^s\mathcal{S}^* \otimes S^{d-s}W^*$, where $\mathcal{S} = \mathcal{O}_{\mathbb{P}W}(-1)$ is the tautological subbundle over $\mathbb{P}W$. The higher cohomology of $Sym(\eta)$ vanishes. Theorem 6.2.1 implies that the normalization of the coordinate ring of $F_s(S^dW)$ has the decomposition

$$\mathrm{Nor}(\mathbb{C}[F_s(S^dW)])_e = S^{es}W^* \otimes S^e(S^{d-s}W^*).$$

This decomposition implies that $\mathbb{C}[F_s(S^dW)]$ is nonnormal because $\mathbb{C}[F_s(S^dW)]_1 = S^dW^*$ and $\mathrm{Nor}(\mathbb{C}[F_s(S^dW)])_1 = S^sW^* \otimes S^{d-s}W^*$, but on the other hand, if X is a normal, affine variety and $f\colon Y \to X$ is a desingularization, then $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$. Thus, to determine $\mathbb{C}[F_s(S^dW)]$ one would need to deal with the nonnormality of $F_s(S^dW)$. However, in the situation of the proof of Theorem 6.1.4 it is possible to partially avoid such issues.

7. Orbits and their closures.

7.1. Comparing $\overline{GL_{n^2} \cdot \det_n}$ and $End(\mathbb{C}^{n^2}) \cdot \det_n$. In this section we compare the orbit closure $\overline{GL(W) \cdot \det_n}$ with the orbit $GL(W) \cdot \det_n$ and the set $End(W) \cdot \det_n$. The reasons for the first comparison have been discussed already—the second comparison could be useful for helping to understand the first, and it is also important because Valiant's conjecture is related to $End(W) \cdot \det_n$.

In our preprint from July of 2009 we asked if one had the equality $\overline{GL(W) \cdot \det_n} = End(W) \cdot \det_n$. Since then, it has been shown that the equality fails; see [35, Prop. 3.5.1].

A method to construct polynomials belonging to $\overline{GL(W)\cdot \det_n}$ but not to $End(W)\cdot \det_n$ is proposed in [45, pp. 508–510]. The idea is to start from a weighted graph G with n (ordered) vertices, with n even. Consider its skew-adjacency matrix M_G, the skew-symmetric matrix whose (i,j)-entry with $i<j$ is a variable y_{ij} if there is an edge between the vertices i and j, and zero otherwise. More generally, define $M_G(t)$ as before but replacing y_{ij} by $t^{w_{ij}}y_{ij}$, where $w_{ij}\in\mathbb{Z}_{>0}$ denotes the weight of the edge ij. Then

$$\det(M_G(t)) = [\text{Pfaff } M_G(t)]^2 = t^{2W}h_G(y) + \text{higher order terms},$$

where W is the minimal weight of a perfect matching of G, and $h_G(y)$ is a sum of monomials indexed by pairs of minimal perfect matchings. By construction, the polynomial $h_G(y)$ is in $\overline{GL(W)\cdot \det_n}$. In general, G has a unique minimal perfect matching, so $h_G(y)$ is just a monomial which belongs to $End(W)\cdot\det_n$. It is conjectured in [45, sect. 4.2] that there exist pathological weighted graphs G such that $h_G(y)$ does not have a small size formula and does not belong to $End(W)\cdot\det_n$.

7.2. Towards understanding $\overline{GL(W)\cdot \det_n}\subset S^nW$. In order to better understand the coordinate ring of $\overline{GL(W)\cdot \det_n}$, it will be important to answer the following question.

QUESTION 7.2.1. *What are the irreducible components of the boundary of the orbit closure $\overline{GL(W)\cdot \det_n}$? Are they $GL(W)$-orbit closures?*

In principle, $\overline{GL(W)\cdot \det_n}$ can be analyzed as follows. The action of $GL(W)$ or $End(W)$ on $\det_n$ defines a rational map

$$\psi_n : \mathbb{P}(End(W)) \dashrightarrow \mathbb{P}(S^nW^*)$$

given by $[u]\mapsto[\det_n\circ u]$. Its indeterminacy locus $I(\psi_n)$ is, set theoretically, given by the set of u such that $\det(u.X)=0$ for all $X\in W=\mathrm{Mat}_{n\times n}$. Thus

$$I(\psi_n)=\{u\in End(W) \mid \ \mathrm{Im}(u)\subset \mathrm{Det}_n\},$$

where $\mathrm{Det}_n\subset W$ denotes the hypersurface of noninvertible matrices. Since $\mathrm{Im}(u)$ is a vector space, this relates the problem of understanding ψ_n to that of linear subspaces in the determinantal hypersurface $\mathbb{P}(\mathrm{Det}_n)\subset\mathbb{P}(End(W))$, which has already received some attention (see, e.g., [16]).

By Hironaka's theorems [25] one can resolve the indeterminacy locus of ψ_n by a sequence of smooth blowups, and $\overline{GL(W)\cdot \det_n}$ can then be obtained as the image of the resolved map. Completely resolving the indeterminacies will probably be too difficult, but this approach should help to answer the preceding questions.

7.3. Remarks on the extension problem. Let G be reductive, let V be an irreducible G-module, and let $v\in V$. Consider the closure $\overline{G\cdot v}$ of the G-orbit $G\cdot v\simeq G/G(v)$. Then the boundary $\overline{G\cdot v}\backslash G\cdot v$ has finitely many components $H_1,\dots,H_N$ of codimension at least one in $\overline{G\cdot v}$. If G is connected, each of these components is a G-variety. Moreover, if $G(v)$ is reductive, then all H_i have codimension one; cf. section 4.2.

Example 7.3.1. The most classical example of all for the extension problem is $\mathbb{C}^*\subset\mathbb{C}$. Here $\mathbb{C}[\mathbb{C}^*]=\mathbb{C}[z,z^{-1}]$ and $\mathbb{C}[\overline{\mathbb{C}^*}]=\mathbb{C}[\mathbb{C}]=\mathbb{C}[z]$. We can take $G=\mathbb{C}^*$ and $v=1$.

Consider the case where the singular locus of $\overline{G\cdot v}$ has codimension at least two. Then the generic point of each codimension one H_i is a smooth point of $\overline{G\cdot v}$, so that

H_i can be defined around that point by a regular function h_i, uniquely defined up to an invertible function. This allows one to define a valuation ν_i on $\mathbb{C}[G \cdot v]$, giving the order of the pole of a rational function along H_i: each regular function f on $G \cdot v$, considered as a rational function of $\overline{G \cdot v}$, can be uniquely written at the generic point of H_i as $f = g h_i^{\nu_i(f)}$, where g is regular and invertible, and $\nu_i(f) \in \mathbb{Z}$. The valuation ν_i is G-invariant if H_i is. Since a regular function on $\overline{G \cdot v}$ has no poles, we have

$$\mathbb{C}[\overline{G \cdot v}] \subset \{f \in \mathbb{C}[G \cdot v] \mid \ \forall i \ \nu_i(f) \geq 0\}.$$

If, moreover, $\overline{G \cdot v}$ is normal, then equality holds: if $f \in \mathbb{C}[G \cdot v]$ is such that $\nu_i(f) \geq 0$ for all i, then f is regular at the generic point of any codimension one boundary component of $\overline{G \cdot v}$ and hence outside a subset of codimension at least two—hence everywhere (see, e.g., [15, Cor. 11.4]). (Earlier, Kostant [31, Prop. 9, p. 351] showed that if the boundary of $\overline{G \cdot v}$ has codimension at least two in $\overline{G \cdot v}$, and $\overline{G \cdot v}$ is normal, then $\mathbb{C}[\overline{G \cdot v}] = \mathbb{C}[G \cdot v]$.)

In July of 2009 we wrote that we expected this normality condition and the codimension two singularities condition to fail in our cases. Since then, Kumar [33] proved that neither the orbit of the determinant nor of the permanent is a normal variety. Nevertheless, the analysis of codimension one boundary components of the orbit $G \cdot v$ should be a first step towards the determination of $\mathbb{C}[\overline{G \cdot v}]$. We also point out that the boundary of the orbits of the permanent and determinant are of pure codimension one, as their stabilizers are reductive; cf. section 4.2.

Another instance of an extension problem was the problem essentially solved by Demazure (with corrections made by Kac, Joseph, and others; see the comments in section 8.3.C of [34]) for B-orbits in G/B, where G is semisimple and $B \subset G$ is a Borel subgroup. Here the orbits, which are Schubert cells, are just affine spaces (and thus have very simple coordinate rings) and the closures are Schubert varieties. For a precise, more general statement and references, see [34, Thm. 8.2.2]. This result relies on the normality of the Schubert varieties, which, as remarked above, fails for the orbit closures of interest here.

Finally, we remark that a recent work [13] tries to apply the GCT approach to the problem of proving lower bounds on tensor rank. One of the main outcomes of this work is that by looking at SL-obstructions only trivial lower bounds can be shown.

8. Kronecker coefficients. We have seen that we need to understand the Kronecker coefficients $k_{\delta^n,\delta^n,\pi}$ in order to understand $\mathbb{C}[GL(W) \cdot \det_n]$. Similarly, in order to understand $\mathbb{C}[GL(W) \cdot \ell^{n-m}\mathrm{perm}_m]$ we need to understand Kronecker coefficients $k_{\pi\mu\nu}$ where $S_\mu\mathbb{C}^m$ and $S_\nu\mathbb{C}^m$ are contained in some plethysm $S^m(S^k\mathbb{C}^m)$. We first give general facts about computing Kronecker coefficients which tell us the multiplicities of certain modules in the coordinate rings we are interested in. Since keeping track of the multiplicities in the cases at hand appears to be hopeless, one could try to solve the simpler question of nonvanishing of Kronecker coefficients (i.e., that a certain module appears at all), so we next discuss conditions where one can determine whether Kronecker coefficients are nonzero. Finally in the last two subsections we specialize to the types of Kronecker coefficients arising in the study of $\mathbb{C}[\det_n]$ and $\mathbb{C}[\ell^{n-m}\mathrm{perm}_m]$.

8.1. General facts. A general reference for this section is [36, sect. I.7]. Let π, μ, ν be three partitions of a number n. The Kronecker coefficient $k_{\pi\mu\nu}$ is the dimension of the space of $\mathfrak{S}_n$-invariants in $[\pi] \otimes [\mu] \otimes [\nu]$, where recall that $[\pi]$ is the irreducible $\mathfrak{S}_n$-module associated with π. In particular, $k_{\pi\mu\nu}$ is *symmetric* with respect to π, μ, ν. Since the irreducible complex representations of $\mathfrak{S}_n$ are all defined over $\mathbb{Q}$, $k_{\pi\mu\nu}$ is also the multiplicity of $[\pi]$ inside the tensor product $[\mu] \otimes [\nu]$.

Write $\pi = (n - |\bar{\pi}|, \bar{\pi})$. Then $k_{\pi\mu\nu}$ depends only on the triple $(\bar{\pi}, \bar{\mu}, \bar{\nu})$ when n is sufficiently large; cf. [54]. A more precise statement was obtained in [7]. It implies that if $k_{\pi\mu\nu} \neq 0$, then $|\bar{\pi}| \leq |\bar{\mu}| + |\bar{\nu}|$. Moreover, in case of equality, the Kronecker coefficient can be identified with a Littlewood–Richardson coefficient:

$$k_{\pi\mu\nu} = c^{\bar{\pi}}_{\bar{\mu},\bar{\nu}}.$$

Relation with characters. Kronecker coefficients can be computed from the characters of the irreducible representations of $\mathfrak{S}_n$. Let χ_π denote the character of $[\pi]$. Then (see [36, p. 115])

$$k_{\pi\mu\nu} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi_\pi(w)\chi_\mu(w)\chi_\nu(w). \tag{8.1.1}$$

The characters of $\mathfrak{S}_n$ can be computed in many ways. Following the Frobenius character formula, they appear as coefficients of the expansion of Newton symmetric functions p_μ in terms of Schur functions s_π:

$$p_\mu = \sum_\pi \chi^\mu_\pi s_\pi.$$

Here χ^μ_π denotes the value of the character χ_π on any permutation of cycle type μ. Another formula for χ^μ_π is given by the Murnaghan–Nakayama rule, which involves a certain type of tableaux T of shape π and weight μ (that is, numbered in such a way that each integer i appears μ_i times). Call T a *multiribbon tableau* if it is numbered nondecreasingly on each row and column, in such a way that for each i, the set of boxes numbered i forms a ribbon (a connected set containing no two-by-two square). Then

$$\chi^\mu_\pi = \sum_T (-1)^{h(T)},$$

where the sum is over all multiribbon tableaux T of shape π and weight μ, and $h(T)$ is the sum of the heights of the ribbons in T (the height of a ribbon being the number of rows it occupies, minus one); see, e.g., [36, sect. I.7, Ex. 5]).

Small length cases. The symmetric group $\mathfrak{S}_n$ has two one dimensional representations, the trivial representation $[n]$ and the sign representation $[1^n]$. One has

$$[n] \otimes [\pi] = [\pi] \qquad \text{and} \qquad [1^n] \otimes [\pi] = [\pi^*],$$

where π^* denotes the conjugate partition of π. After these two, the simplest representation of $\mathfrak{S}_n$ is the vector representation $[n-1, 1]$ on n-tuples of complex numbers with sum zero. Its exterior powers $\wedge^p[n-1, 1] = [n-p, 1^p]$ are irreducible. Recently Ballantine and Orellana [1] computed the product of $[n-p, p]$ with $[\pi]$ under the condition that $\pi_1 \geq 2p - 1$ (or $\pi^*_1 \geq 2p - 1$).

Schur–Weyl duality. There is a close connection between representations of symmetric groups and representations of general linear groups, called Schur–Weyl duality [26]. Consider the tensor power $U^{\otimes n}$ of a complex vector space U. The diagonal action of $GL(U)$ commutes with the permutation action of $\mathfrak{S}_n$. Schur–Weyl duality is the statement that, as a $GL(U) \times \mathfrak{S}_n$-module,

$$U^{\otimes n} = \bigoplus_{|\pi|=n} S_\pi U \otimes [\pi].$$

A straightforward consequence is the already stated fact that the Kronecker coefficient $k_{\pi\mu\nu}$ can be defined as the multiplicity of $S_\mu V \otimes S_\nu W$ inside $S_\pi(V \otimes W)$ (at least for V and W of large enough dimension). In particular, since $[n]$ is the trivial representation, this yields the Cauchy formula

$$S^n(V \otimes W) = \bigoplus_{|\pi|=n} S_\pi V \otimes S_\pi W.$$

Using the Giambelli formula (which expresses any Schur power in terms of symmetric powers) and the Cauchy formula, it is easy to express any Kronecker coefficient in terms of Littlewood–Richardson coefficients. If π has length ℓ, we denote the multiplicity of $S_\mu V$ in $S_{\alpha_1} V \otimes \cdots \otimes S_{\alpha_\ell} V$ by $c^\mu_{\alpha_1,\ldots,\alpha_\ell}$. Then

$$k_{\pi\mu\nu} = \sum_{w \in \mathfrak{S}_\ell} \operatorname{sgn}(w) \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell),\\ |\alpha_i| = \pi_i - i + w(i)}} c^\mu_{\alpha_1,\ldots,\alpha_\ell} c^\nu_{\alpha_1,\ldots,\alpha_\ell}. \tag{8.1.2}$$

8.2. Nonvanishing of Kronecker coefficients.

The semigroup property. A rephrasing of the Schur–Weyl duality yields the decomposition

$$Sym(U \otimes V \otimes W) = \bigoplus_{\pi,\mu,\nu} (S_\pi U \otimes S_\mu V \otimes S_\nu W)^{\oplus k_{\pi\mu\nu}}. \tag{8.2.1}$$

Using the fact that the highest weight vectors in this algebra form a finitely generated subalgebra, one can deduce (see [14]) the following:

- Triples of partitions with nonzero Kronecker coefficients form a semigroup; that is, if $k_{\pi\mu\nu} \neq 0$ for partitions π, μ, ν of some integer n, and $k_{\pi'\mu'\nu'} \neq 0$ for partitions π', μ', ν' of n', then
$$k_{\pi+\pi',\mu+\mu',\nu+\nu'} \neq 0.$$
- If one restricts to triples of partitions of length bounded by some integer ℓ, the corresponding semigroup is finitely generated.
- If $k_{\pi\mu\nu} \neq 0$, the normalized partitions $\tilde\pi = \frac{\pi}{n}$, $\tilde\mu = \frac{\mu}{n}$, $\tilde\nu = \frac{\nu}{n}$ verify the entropy relations
$$H(\tilde\pi) \leq H(\tilde\mu) + H(\tilde\nu). \tag{8.2.2}$$
Here $H(\tilde\pi) = -\sum_i \tilde\pi_i \log(\tilde\pi_i)$ denotes the *Shannon entropy* [60].

Saturation does not hold for Kronecker coefficients; that is, $k_{N\pi,N\mu,N\nu} \neq 0$ for some $N \geq 2$ does not imply that $k_{\pi,\mu,\nu} \neq 0$. For counterexamples, see [6], whose appendix by Mulmuley contains several conjectures regarding the saturation property.

Linear constraints for vanishing. Consider the set KRON of triples $(\tilde{\pi}, \tilde{\mu}, \tilde{\nu})$, where π, μ, ν are three partitions of n such that $k_{\pi\mu\nu} \neq 0$ and $\tilde{\pi}$, etc., are as above. Let KRON_ℓ denote the analogous set with the additional condition that the length of the three partitions be bounded by ℓ. One can deduce from the previous remarks that KRON_ℓ is a rational convex polytope (see, e.g., [17, 14]).

What are the equations of the facets of this polytope? A geometric method to produce many such facets appears in [39], in terms of embeddings

$$\varphi_T : \mathcal{F}(V) \times \mathcal{F}(W) \hookrightarrow \mathcal{F}(V \otimes W).$$

Here $\mathcal{F}(V)$ (resp., $\mathcal{F}(W)$) denotes the variety of full flags in the vector space V (resp., W) of dimension m (resp., n). There is no canonical way to define a flag H in $V \otimes W$ from a flag F in V and a flag G in W. In order to do that, one needs to prescribe what Klyachko calls a *cubicle*: a numbering T of the boxes (i, j) of a rectangle $m \times n$ by integers $\ell_T(i, j)$ running from 1 to mn, increasingly on each line and column. Then one lets

$$H_k = \varphi_T(F, G)_k = \sum_{\ell_T(i,j) \leq k} F_i \otimes G_j.$$

Klyachko [29] goes one step further by applying results of [2]. To state his result, we need a definition. Consider two nonincreasing sequences a and b of real numbers, of lengths m and n, each of sum zero. Suppose that the real numbers $a_i + b_j$ are all distinct. Ordering them defines a sequence $a + b$ of length nm and thus a cubicle T and the associated map φ_T. Recall that the integral cohomology ring $H^*(\mathcal{F}(V))$ has a natural basis given by the Schubert classes σ_u, indexed by permutations $u \in \mathfrak{S}_m$. For any permutation $w \in \mathfrak{S}_{mn}$, we can therefore decompose the pull-back by φ_T of the corresponding Schubert class as

$$\varphi_T^* \sigma_w = \sum_{\substack{u \in \mathfrak{S}_m \\ v \in \mathfrak{S}_n}} c_{uv}^w(a, b) \sigma_u \otimes \sigma_v.$$

The coefficients $c_{uv}^w(a, b)$ are nonnegative integers. Klyachko's statement is the following.

THEOREM 8.2.1 (see [29]). *Suppose $\ell \geq m, n$. Then $(\tilde{\pi}, \tilde{\mu}, \tilde{\nu})$ belongs to* KRON_ℓ *iff*

$$\sum_i a_i \tilde{\pi}_{u(i)} + \sum_j b_j \tilde{\mu}_{v(j)} \geq \sum_k (a + b)_k \tilde{\nu}_{w(k)}$$

for all nonincreasing sequences a, b and for all $u \in \mathfrak{S}_m, v \in \mathfrak{S}_n, w \in \mathfrak{S}_{mn}$ such that $c_{uv}^w(a, b) \neq 0$.

There is a formula for the coefficients $c_{uv}^w(a, b)$ in terms of divided difference operators, which allows one to make explicit computations in low dimensions. For example, one can recover the description of KRON_3 given by Franz [17] as the convex hull of 11 explicit points. Unfortunately, there is no general rule for deciding whether or not $c_{uv}^w(a, b)$ is zero. Moreover, the number of inequalities seems to grow extremely fast with ℓ. Redundancy is also an issue. Klyachko conjectures that it is enough, as for the Horn problem, to consider inequalities for which $c_{uv}^w(a, b) = 1$. Recent advances by Ressayre [58] allow one, in principle, to get a complete and irredundant list of facets for KRON_ℓ.

In [11] the set of $(\tilde{\pi}, \tilde{\mu}, \tilde{\nu}) \in$ KRON with the additional condition that $\tilde{\mu}, \tilde{\nu}$ be the uniform distributions of length ℓ were studied. The resulting $\tilde{\pi}$ can be *any* probability distribution on ℓ^2 points so that the containment in KRON does not impose any constraint. This is significant in view of Proposition 5.2.1 and shows that "candidates" for obstructions are in a sense rare.

8.3. Case of rectangular partitions.

Stanley's character formula. Formula (8.1.1) shows that, in order to compute a Kronecker coefficient of type $k_{\delta^n,\delta^n,\pi}$, it would be useful to have a nice formula for the character χ_{δ^n}. Recall that δ^n denotes the partition whose diagram is a rectangle $\delta \times n$ (i.e., the partition $(\delta, \dots, \delta) = (\delta^n)$). Such a formula is given by Stanley in [63]. Suppose that w is a permutation in $\mathfrak{S}_{\delta n}$. Then

$$\chi_{\delta^n}(w) = \frac{(-1)^{\delta n}}{\prod_{i=1}^{\delta}\prod_{j=1}^{n}(i+j-1)} \sum_{uv=w} \delta^{\kappa(u)}(-n)^{\kappa(v)},$$

where $u, v \in \mathfrak{S}_{\delta n}$ and $\kappa(u)$ denotes the number of cycles in u.

Relations with invariants. Let U, V, W be vector spaces of dimensions ℓ, n, n, respectively. Taking $SL(V) \times SL(W)$-invariants in formula (8.2.1) yields

$$A := Sym(U \otimes V \otimes W)^{SL(V)\times SL(W)} = \bigoplus_{\delta,\pi} (S_\pi U)^{\oplus k_{\pi,\delta^n,\delta^n}}.$$

For $\ell = 2$ it is known that $A \simeq Sym(S^n U)$ [61, Thm. 17, p. 369]. Thus for a partition $\pi = (a, b)$ of δn in two parts, $k_{\pi,\delta^n,\delta^n}$ is equal to the multiplicity of $S_\pi U$ in $S^\delta(S^n U)$. This is given by *Sylvester's formula* (see, e.g., [62, Thm. 3.3.4]):

$$k_{(\delta n-b,b),\delta^n,\delta^n} = P(b; \delta \times n) - P(b-1; \delta \times n), \tag{8.3.1}$$

where $P(b; \delta \times n)$ denotes the number of partitions of size b inside the rectangle $\delta \times n$.

This also follows directly from formula (8.1.2), once we observe that a Littlewood–Richardson coefficient $c^{\delta^n}_{\alpha,\beta}$ is nonzero only if α and β are complementary partitions in the rectangle $\delta \times n$, and in that case it equals one (this is a straightforward consequence of the Littlewood–Richardson rule and a version of Poincaré duality for Grassmannians).

The same argument yields a formula for the length three case as follows. Let $\pi = (a, b, c)$ with $a+b+c = \delta n$. Denote by $ST(a, b; \delta \times n)$ the number of semistandard lattice permutation skew-tableaux whose shape is of the form β/α, for β a partition of size $\delta n - b$ in the rectangle $\delta \times n$, and α a partition of size a (see [36] for the terminology). Then

$$\begin{aligned} k_{\pi,\delta^n,\delta^n} &= ST(a, b; \delta \times n) - ST(a, b+1; \delta \times n) + ST(a+1, b+1; \delta \times n) \\ &\quad - ST(a+1, b-1; \delta \times n) + ST(a+2, b-1; \delta \times n) - ST(a+2, b; \delta \times n). \end{aligned}$$

For $n = 2$, and $\dim U = 4$, the algebra of highest weight vectors in A turns out to be polynomial, with generators of weights $(2), (22), (222)$, and (1111) [41]. Call a partition even (resp., odd) if all its parts are even (resp., odd). We deduce the following.

PROPOSITION 8.3.1. *A Kronecker coefficient $k_{\pi,(\delta\delta),(\delta\delta)}$ is nonzero iff*

- *either π is an even partition of 2δ, of length at most four, or*
- *π is an odd partition of 2δ, of length exactly four.*

In both cases $k_{\pi,(\delta\delta),(\delta\delta)} = 1$.

Constraints. Let $[\pi]$ be a component of $[(\delta^n)] \otimes [(\delta^n)]$. The entropy relations (8.2.2) yield

$$H(\tilde{\pi}) \leq 2\log(n).$$

Denote $|\pi|_{\leq a} = \pi_1 + \cdots + \pi_a$ (and similarly $|\pi|_{\geq a}$, etc., ...). Then [39, Théorème 3.2] gives

$$|\pi|_{>ab} \leq \delta(n-a)^+ + \delta(n-b)^+,$$

where $x^+ = x$ if x is positive and zero otherwise. For example, $|\pi|_{\leq n} \geq \delta$.

8.4. A variant of Schur–Weyl duality. By Schur–Weyl duality, the decomposition of the Schur powers $S_\pi(V_1 \otimes \cdots \otimes V_m)$ into irreducible components, for $|\pi| = \ell$, is equivalent to the decomposition of tensor products of m irreducible representations of $\mathfrak{S}_\ell$. What happens if we let $V_1 = \cdots = V_m = V$ and replace the tensor product $V_1 \otimes \cdots \otimes V_m$ by the mth symmetric power of V?

The following remarkable theorem is proved in [21]. Suppose V has dimension n, and fix a basis of V. This defines an action of $\mathfrak{S}_n$ on V and on any Schur power $S_\mu V$. In particular, the zero weight space $(S_\mu V)_0$ is an $\mathfrak{S}_n$-module, nontrivial iff μ is of size $n\delta$ for some δ. Here zero weight must be understood with respect to a maximal torus in $SL(V)$.

THEOREM 8.4.1 (see [21]). *Let* $\dim V = n$ *and let* μ *be a partition of* $n\delta$ *(so that* $(S_\mu V)_0 \neq 0$*). Suppose that the decomposition of* $(S_\mu V)_0$ *into irreducible* $\mathfrak{S}_n$*-modules is*

$$(S_\mu V)_0 = \bigoplus_\pi [\pi]^{\oplus s_{\mu,\pi}}.$$

Then one has the decomposition of $GL(V)$*-modules*

$$S_\pi(S^\delta V) = \bigoplus_\mu (S_\mu V)^{\oplus s_{\mu,\pi}}.$$

In particular, for $\delta = 1$, i.e., $|\mu| = n$, we have $(S_\mu V)_0 = [\mu]$.

COROLLARY 8.4.2. *Let* μ *be a partition of size* $n\delta$*. The dimension of the space of* $\mathfrak{S}_n$*-invariants in the zero weight space* $(S_\mu \mathbb{C}^n)_0$ *equals the multiplicity of* $S_\mu \mathbb{C}^n$ *in the plethysm* $S^n(S^\delta \mathbb{C}^n)$*.*

For $\delta = 2$, because of the formula [36, Ex. 6(a), p. 138], this implies that $(S_\mu V)_0$ contains nontrivial $\mathfrak{S}_n$-invariants iff μ is even. For general δ, conditions for multiplicities to not vanish have been obtained in [7, 40]. Recently, in response to our paper, it was shown in [12] that whenever δ is even and all the parts μ_i are of even size, then $S_\mu \mathbb{C}^n$ occurs in $S^n(S^\delta \mathbb{C}^n)$. Hence $(S_\mu \mathbb{C}^n)_0$ contains $\mathfrak{S}_n$-invariants in this case.

Observe that for $n = \dim V = 2$, these multiplicities are given by Sylvester's formula (8.3.1). This can be generalized as follows. Consider a finite dimensional $GL(V)$-module M, and let $m_\mu(M)$ denote the multiplicity of the weight μ in M. Let $N_\pi(M)$ denote the multiplicity of $S_\pi V$ in the decomposition of M into irreducible components. Then

$$N_\pi(M) = \sum_{w \in \mathfrak{S}_n} \operatorname{sgn}(w) m_{w(\pi+\rho)-\rho}(M), \tag{8.4.1}$$

where $\rho = (n, \ldots, 2, 1)$. Indeed, the Weyl character formula is equivalent to (8.4.1) when M is irreducible. By linearity, it must hold for any M. In particular, let

$M = S^n(S^\delta V)$. The multiplicity $m_\mu(M)$ is then equal to the number $p(\mu; n, \delta)$ of ways of writing the monomial x^μ as a product of n monomials of degree δ. The multiplicity of $S_\pi V$ inside $S^n(S^\delta V)$ is thus

$$N(\pi; n, \delta) = \sum_{w \in \mathfrak{S}_n} \operatorname{sgn}(w) p(w(\pi + \rho) - \rho; n, \delta),$$

which generalizes Sylvester's formula.

9. Complexity classes. In this section we explain the precise complexity problem studied by the GCT program, namely $\overline{\mathbf{VP}_{\mathrm{ws}}} \neq \mathbf{VNP}$, and place it in the context of Valiant's algebraic model of NP-completeness [66, 65]. In particular, we compare this to the conjecture $\mathbf{VP} \neq \mathbf{VNP}$, and that the permanent is not a p-projection of the determinant, the latter being equivalent to the conjecture $\mathbf{VP}_{\mathrm{ws}} \neq \mathbf{VNP}$. The conjecture $\mathbf{VP} \neq \mathbf{VNP}$ is an arithmetic analogue of the conjecture $\mathbf{P} \neq \mathbf{NC}$.

All polynomials considered are over $\mathbb{C}$. A general reference for this section is [8].

9.1. Models of arithmetic circuits and complexity. An *arithmetic circuit* is a finite acyclic directed graph with vertices of in-degree 0 or 2 and exactly one vertex of out-degree 0. Vertices of in-degree 0 are called *inputs* and labeled by a constant in $\mathbb{C}$ or a variable. The other vertices, of in-degree 2, are labeled by $\times$ or $+$ and called *computation gates*. We define the *size* of a circuit as the number of its vertices. The *depth* of the circuit is defined as the maximum length of a directed path in the underlying graph. The polynomial computed by a circuit is easily defined by induction.

If the graph underlying the circuit is a directed tree, i.e., all vertices have out-degree at most 1, then we call the circuit an *expression* or *formula*. The notion of *weakly skew* circuits is less restrictive: we require that for each multiplication gate α, at least one of the two vertices pointing to α be computed by a separate subcircuit C_α. Separate means that the edge connecting C_α to α is the only edge between a vertex of C_α and the remainder to the circuit. In short, formulas are circuits where previously computed values cannot be reused, while in weakly skew circuits we require that at least one of the two operands of a multiplication gate be computed just for that gate. We note that the degree of the polynomial computed by a weakly skew circuit is bounded by its size. The motivation for weakly skew circuits is that they exactly characterize the determinant, as we explain below.

We define the *complexity* $L(f)$ of a polynomial f over $\mathbb{C}$ as the minimum size of an arithmetic circuit computing f. Restricting to weakly skew circuits and formulas, respectively, one defines the corresponding complexity notions $L_{\mathrm{ws}}(f)$ and $L_e(f)$. Clearly, $L_e(f) \geq L_{\mathrm{ws}}(f) \geq L(f)$. The quantity $L_e(f)$ is called the *formula size* of f. It is an important fact [5] that $\log L_e(f)$ equals, up to a constant factor, the minimum depth of an arithmetic circuit computing f.

An algorithm due to Berkowitz [3] for computing the determinant implies that $L_{\mathrm{ws}}(\det_n) = \mathcal{O}(n^5)$. This algorithm also shows the well-known fact $\log(L_e(\det_n)) = \mathcal{O}(\log^2 n)$. The best known upper bound $L(\mathrm{per}_m) = \mathcal{O}(m 2^m)$ on the complexity of the permanent is exponential [59].

The complexity class $\mathbf{VP}_e$ is defined as the set of sequences (f_n) of multivariate polynomials over $\mathbb{C}$ such that $L_e(f_n)$ is polynomially bounded in n. The set of sequences (f_n) such that $L_{\mathrm{ws}}(f_n)$ is polynomially bounded in n comprises the complexity class $\mathbf{VP}_{\mathrm{ws}}$. The class $\mathbf{VP}$ is defined as the the set of sequences (f_n) such that $L(f_n)$ and $\deg f_n$ are polynomially bounded in n (it is possible to give a syntactic characterization of **VP** in terms of multiplicatively disjoint circuits [38]). Note that $\mathbf{VP}_e \subseteq \mathbf{VP}_{\rm ws} \subseteq \mathbf{VP}$. Since $L_{\rm ws}(\det_n) = \mathcal{O}(n^5)$, we have $(\det_n) \in \mathbf{VP}_{\rm ws}$. It is a major open question whether $(\det_n)$ is contained in $\mathbf{VP}_e$. This is equivalent to the question of whether $\det_n$ can be computed by arithmetic circuits of depth $\mathcal{O}(\log n)$. The best known upper bound is $\mathcal{O}(\log^2 n)$; see [3].

9.2. Completeness. A polynomial f is called a *projection* of a polynomial g if f can be obtained from g by substitution of the variables by variables or constants. A sequence (f_n) is called a *p-projection* of a sequence (g_n) if there exists a polynomially bounded function $t\colon \mathbb{N} \to \mathbb{N}$ such that f_n is a projection of $g_{t(n)}$ for all n. We note that each of the previously introduced complexity classes $\mathcal{C}$ is closed under p-projection; i.e., if (f_n) is p-projection of (g_n) and $(g_n) \in \mathcal{C}$, then $(f_n) \in \mathcal{C}$. A sequence (g_n) is called $\mathcal{C}$-*complete* iff $(g_n) \in \mathcal{C}$ and any $(f_n) \in \mathcal{C}$ is a p-projection of (g_n).

The determinant has the following important universality property [66, 64, 38]: if $L_{\rm ws}(f) \le m$, then f is a projection of $\det_{m+1}$. This implies that the sequence $(\det_n)$ of determinants is $\mathbf{VP}_{\rm ws}$-complete [64]. Therefore, $\mathbf{VP}_e = \mathbf{VP}_{\rm ws}$ is equivalent to $(\det_n) \in \mathbf{VP}_e$, the major open question mentioned before. It is not known whether $\mathbf{VP}_{\rm ws}$ is different from **VP**.

We remark that when replacing polynomial upper bounds by quasi-polynomial upper bounds $2^{\log^c n}$ in the definitions of the above three complexity classes, then all these classes coincide.

We assign now to any of the above complexity classes $\mathbf{VP}_?$ a corresponding "nondeterministic" complexity class $\mathbf{VNP}_?$ as follows. A sequence (f_n) of polynomials belongs to $\mathbf{VNP}_?$ if there exist a polynomial p and a sequence $(g_n) \in \mathbf{VP}_?$ such that $f_n(x) = \sum_e g_n(x,e)$ for all n, where the sum is over all $e \in \{0,1\}^{p(n)}$. It is a nontrivial fact that the resulting classes are the same: $\mathbf{VNP}_e = \mathbf{VNP}_{\rm ws} = \mathbf{VNP}$; for an intuitive proof, see [38]. Clearly $\mathbf{VP} \subseteq \mathbf{VNP}$.

Valiant [66] proved the major result that (per_n) is **VNP**-complete. Thus $(\mathrm{per}_n) \notin \mathbf{VP}$ is equivalent to $\mathbf{VP} \neq \mathbf{VNP}$, which is sometimes called *Valiant's hypothesis*. This can be seen as an algebraic version of Cook's famous $\mathbf{P} \neq \mathbf{NP}$ hypothesis. There is great empirical evidence that Valiant's hypothesis is true: if it were false, then most of the complexity classes considered by researchers today would collapse [9]. Proving this implication relies on the generalized Riemann hypothesis, but we note that the latter can be omitted when dealing with the constant-free versions of the complexity classes (where only $0, 1$ are allowed as constants instead of any complex numbers).

It is natural to weaken Valiant's hypothesis to $\mathbf{VP}_{\rm ws} \neq \mathbf{VNP}$. In view of the completeness of the sequences of determinants and permanents in $\mathbf{VP}_{\rm ws}$ and **VNP**, respectively, $\mathbf{VP}_{\rm ws} \neq \mathbf{VNP}$ is logically equivalent to the claim that (per_n) is not a p-projection of $(\det_n)$. The latter is a purely mathematical statement, not involving any notions of computation. This is why some people (including ourselves) believe that this offers one of the most promising possibilities to attack the **P** vs. **NP** problem.

9.3. Approximate complexity classes. In [10] it was proposed to study the notion of approximate complexity in Valiant's framework. There is a natural way to put a topology on the polynomial ring $A := \mathbb{C}[X_1, X_2, \ldots]$ as a limit of the Euclidean topologies on the finite dimensional subspaces $\{f \in \mathbb{C}[X_1, \ldots, X_n] \mid \deg f \le d\}$ whose union over n, d is A.

DEFINITION 9.3.1. *The* approximate complexity $\underline{L}(f)$ *of* $f \in A$ *is defined as the minimum* $r \in \mathbb{N}$ *such that* f *is in the closure of* $\{g \in A \mid L(g) \le r\}$. *Replacing here* $L(g)$ *by* $L_{\rm ws}(g)$ *we obtain the* approximate complexity $\underline{L}_{\rm ws}(f)$.

We remark that the same complexity notions are obtained when using the Zariski topology, since constructible sets have the same closure with respect to Euclidean and Zariski topologies. For more information on approximate complexity, we refer the reader to [10].

We define the complexity class $\overline{\mathbf{VP}}_{\mathrm{ws}}$ as the set of sequences (f_n) of complex polynomials such that $\underline{L}_{\mathrm{ws}}(f_n)$ is polynomially bounded in n. Similarly, one defines the classes $\overline{\mathbf{VP}}$. Clearly, $\overline{\mathbf{VP}}_{\mathrm{ws}} \subseteq \overline{\mathbf{VP}}$ and both classes are closed under p-projections. It is not known whether or not $\overline{\mathbf{VP}}_{\mathrm{ws}}$ is contained in $\mathbf{VNP}$.

We go now back to the GCT approach of [45], which attempts to show Conjecture 1.

PROPOSITION 9.3.2. *Conjecture* 1 *is equivalent to* $(\mathrm{per}_m) \notin \overline{\mathbf{VP}}_{\mathrm{ws}}$ *and equivalent to* $\mathbf{VNP} \not\subseteq \overline{\mathbf{VP}}_{\mathrm{ws}}$.

Before giving the proof we note that Conjecture 1 would imply that $\mathbf{VP}_{\mathrm{ws}} \neq \mathbf{VNP}$ (but not a priori $\mathbf{VP} \neq \mathbf{VNP}$).

Proof. The second equivalence is a consequence of the **VNP** completeness of (per_m). To show the first equivalence suppose first that Conjecture 1 is false. Then there exist $c \geq 1$ and m_0 such that for all $m \geq m_0$, $[\ell^{m^c-m}\mathrm{per}_m]$ is contained in the projective orbit closure $\overline{GL_{m^{2c}} \cdot [\det_{m^c}]}$ in $\mathbb{P}(S^{m^c}\mathbb{C}^{m^{2c}})$. This implies that $\ell^{m^c-m}\mathrm{per}_m \in \overline{GL_{m^{2c}} \cdot \det_{m^c}} \subset S^{n^c}\mathbb{C}^{m^{2c}}$. Thus for fixed $m \geq m_0$, there exists a sequence (σ_k) in $GL_{m^{2c}}$ such that $f_k := \sigma_k \cdot \det_{m^c}$ satisfies $\lim_{k\to\infty} f_k = \ell^{m^c-m}\mathrm{per}_m$. There is a weakly skew arithmetic circuit for $\det_{m^c}$ of size polynomial in m. Composing this circuit with an arithmetic circuit for matrix-vector multiplication that computes the linear transformation σ_k yields a weakly skew arithmetic circuit for f_k of size at most $m^{c'}$, where c' denotes a constant (independent of m, k). (In order to preserve the weak skewness we may need several copies of the circuit computing the linear transformation σ_k.) Let f'_k denote the polynomial obtained from f_k after substituting ℓ by 1 and leaving the variables of per_m unchanged. Then $L_{\mathrm{ws}}(f'_k) \leq L_{\mathrm{ws}}(f_k) \leq m^{c'}$ and $\lim_{k\to\infty} f'_k = \mathrm{per}_m$. Hence, by definition, we have $\underline{L}_{\mathrm{ws}}(\mathrm{per}_m) \leq m^{c'}$ for all $m \geq m_0$, which implies that $(\mathrm{per}_m) \in \overline{\mathbf{VP}}_{\mathrm{ws}}$.

To show the other direction suppose that $(\mathrm{per}_m) \in \overline{\mathbf{VP}}_{\mathrm{ws}}$. Hence there exist $c \geq 1$ and m_0 such that $\underline{L}_{\mathrm{ws}}(\mathrm{per}_m) < m^c$ for all $m \geq m_0$. Fix $m \geq m_0$ and put $n = m^c$ to ease notation. By definition, there exists a sequence of forms f_k such that $\lim_{k\to\infty} f_k = \mathrm{per}_m$ and $L_{\mathrm{ws}}(f_k) < n$ for all k. The universality of the determinant implies that f_k is a projection of $\det_n$, say $f_k(x) = \det(M_k)$, where M_k is an n by n matrix whose entries are affine linear forms in the variables x_i. We homogenize now with respect to an additional variable ℓ; i.e., we substitute x_i by x_i/ℓ and multiply the result by ℓ^n. This implies

$$\ell^{n-m} f_k(x) = \ell^n f_k\left(\frac{1}{\ell}x\right) = \det(M'_k)$$

with a matrix M'_k whose entries are linear forms in x_i and ℓ. Since GL_{n^2} is dense in $\mathrm{Mat}_{n\times n}$, we conclude that the form $\ell^{n-m}f_k$ lies in the closure of $GL_{n^2} \cdot \det_n$. As $\lim_{k\to\infty} f_k = \mathrm{per}_m$, this implies that $\ell^{n-m}\mathrm{per}_m$ lies in the closure of $\det_n$. This holds for all $m \geq m_0$ with $n = m^c$, so Conjecture 1 would be false. ∎

Remark 9.3.3. Using the known fact $L_e(\mathrm{per}_m) = O(m^2 2^m)$ from [59], the proof of Proposition 9.3.2 implies that $\overline{GL_{n^2} \cdot [\ell^{n-m}\mathrm{perm}_m]} \subset \overline{GL_{n^2} \cdot [\det_n]}$ for $n = O(m^2 2^m)$.

9.4. Order of approximation. We now discuss whether approximation is actually necessary. Let $R = \mathbb{C}[[\epsilon]]$ the ring of formal power series in ϵ and let K be its quotient field. Substituting ϵ by 0 defines the morphism $R \to \mathbb{C}, r \mapsto (r)_{\epsilon=0}$ which extends to $S^n R^N \to S^n \mathbb{C}^N$. Note that the group $GL_N(K)$ operates on the scalar extension $S^n K^N$ in the natural way.

The following result is due to Hilbert [24]. For a proof, we refer the reader to Kraft [32, Chap. III, sect. 2.3, Lem. 1].

LEMMA 9.4.1. *Suppose that f lies in the $GL_N(\mathbb{C})$-orbit closure of $g \in S^n \mathbb{C}^N$. Then there exists $\sigma \in GL_N(K)$ such that $F := \sigma \cdot g \in S^n R^N$ satisfies $(F)_{\epsilon=0} = f$.*

Assume we are in the situation of the lemma. By multiplying with a sufficiently high power of ϵ, we get R-linear forms $y_1, \ldots, y_N$ such that

$$g(y_1, \ldots, y_N) = \epsilon^q f + \epsilon^{q+1} \tilde{F} \tag{9.4.1}$$

with some $q \in \mathbb{N}$ and $\tilde{F} \in S^n R^N$. We then say that f can be approximated with order at most q along a curve in the orbit of $\det_n$.

QUESTION 9.4.2. *Suppose that f lies in the orbit closure of* $\det_n$ *in $S^n \mathbb{C}^{n^2}$. Can the order of approximation of f along a curve in the orbit of* $\det_n$ *be bounded by a polynomial in n?*

In [10, Thm. 5.7] an exponential upper bound on the order of approximation is proved in a more general situation.

We show now that if Question 9.4.2 has an affirmative answer, then approximations can be eliminated in the context of the GCT approach.

PROPOSITION 9.4.3. *If Question* 9.4.2 *has an affirmative answer, then we have* $\mathbf{VP}_{\mathrm{ws}} = \overline{\mathbf{VP}}_{\mathrm{ws}}$.

In the present form, this observation is new, although the proof is similar to the arguments in [10]. We make some preparations for the proof. A *skew* arithmetic circuit is an arithmetic circuit such that for each multiplication gate α at least one of the two vertices pointing to α is an input vertex. Hence the multiplication is either by a variable or a constant. It is clear that skew circuits are weakly skew. Astonishingly, skew circuits are no less powerful than weakly skew circuits. For each weakly skew circuit there exists a skew circuit with at most double size that computes the same polynomial; cf. [27].

Let $R = \mathbb{C}[[\epsilon]]$ and $F \in R[X_1, \ldots, X_N]$. We denote by $L_{\mathrm{ws}}(F)$ the smallest size of a weakly skew arithmetic circuit computing F from the variables X_i and constants in R. Write $F = \sum_i f_i \epsilon^i$ with $f_i \in \mathbb{C}[X_1, \ldots, X_N]$.

LEMMA 9.4.4. *We have $L_{\mathrm{ws}}(f_0, \ldots, f_q) = \mathcal{O}(q^2 L_{\mathrm{ws}}(F))$ for any $q \in \mathbb{N}$.*

Proof. Suppose we have a weakly skew circuit of size s computing F from the variables and constants $c = \sum_i c_i \epsilon^i \in R$. By the previous comment we can assume without loss of generality that the circuit is skew. Let $g \in R[X_1, \ldots, X_N]$ be an intermediate result of the computation, and write $g = \sum_i g_i \epsilon^i$ with $g_i \in \mathbb{C}[X_1, \ldots, X_N]$. The idea is to construct an arithmetic circuit that instead of g computes the coefficients $g_0, \ldots, g_q$ up to degree q from the variables and the coefficients $c_0, \ldots, c_q$ of the constants c. This is achieved by replacing each addition of the original circuit by $q+1$ additions of the corresponding coefficients. Each multiplication $f = g \cdot h$ of the original circuit is replaced by $\mathcal{O}(q^2)$ arithmetic operations following $f_k = \sum_{i=0}^k g_i \cdot h_{k-i}$. This results in a circuit of size $\mathcal{O}(sq^2)$. Since the original circuit is assumed to be skew, it is clear that the new circuit can be realized by a skew circuit as well. (We note that it is not obvious how to preserve weak skewness.) ∎

Proof of Proposition 9.4.3. Suppose that $(f_m) \in \overline{\mathbf{VP}}_{\mathrm{ws}}$. Then $\underline{L}_{\mathrm{ws}}(f_m) < n$ with n polynomially bounded in m. Hence f_m is in the closure of the set of polynomials g satisfying $L_{\mathrm{ws}}(g) < n$. By the universality of the determinant, those polynomials g

are projections of $\det_n$ and hence contained in $\overline{GL_{n^2} \cdot \det_n}$. It follows that $f_m \in \overline{GL_{n^2} \cdot \det_n}$. If Question 9.4.2 has an affirmative answer, then f_m can be approximated with order at most q along a curve in the orbit of $\det_n$, where q is polynomially bounded in n and hence in m. Hence we are in the situation (9.4.1) and have

$$F := \det_n(y_1, \dots, y_{n^2}) = \epsilon^q f_m + \epsilon^{q+1} \tilde{F}$$

with R-linear forms $y_1, \dots, y_{n^2}$ in the variables x_{ij} and some polynomial $\tilde{F}$ over R in x_{ij}. From this we conclude that $L_{\mathrm{ws}}(F) = m^{\mathcal{O}(1)}$. Lemma 9.4.4 tells us that $L_{\mathrm{ws}}(f_m) = \mathcal{O}(q^2 L_{\mathrm{ws}}(F))$. Since q was assumed to be polynomially bounded in m, we conclude that $L_{\mathrm{ws}}(f_m)$ is polynomially bounded in m as well. This implies $(f_m) \in \mathbf{VP}_{\mathrm{ws}}$. ∎

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