

RANKS OF TENSORS AND A GENERALIZATION OF SECANT VARIETIES

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ABSTRACT. We investigate differences between X -rank and X -border rank, focusing on the cases of tensors and partially symmetric tensors. As an aid to our study, and as an object of interest in its own right, we define notions of X -rank and border rank for a linear subspace. Results include determining and bounding the maximum X -rank of points in several cases of interest.

1. INTRODUCTION

A central problem in many areas (signal processing, algebraic statistics, complexity theory etc., see e.g., [4, 8, 21, 16]) is to understand the ranks and border ranks of 3-way tensors. Let A, B, C be complex vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and let $p \in A \otimes B \otimes C$. One would like to express p as a sum of decomposable tensors, just as one diagonalizes matrices in order to study them better. A first question is how many decomposable tensors one needs. This question has been answered for generic p and most $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, see [1] for the state of the art. The next question is what is the maximum number of decomposable tensors needed for an arbitrary p , and this question is wide open. One would also like to have normal forms when possible, and tests to see if p can be written as a sum of say r decomposable elements, just as a matrix is the sum of r rank one matrices if all its $(r + 1) \times (r + 1)$ minors vanish. Strassen [22] had the idea to reduce this problem to the study of linear subspaces of spaces of endomorphisms, by considering $p \in A \otimes B \otimes C$ as a linear map $A^* \rightarrow B \otimes C$ and studying the image. Strassen's famous equations for $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n)$ are exactly the expression that, after fixing an identification $C \simeq B^*$ so the image may be thought of as a space of endomorphisms, is that the endomorphisms commute (see [4, 19] for discussions).

In this paper we exploit Strassen's perspective further and generalize it. We determine the possible ranks of 3-way tensors having a given border rank in several cases. In a different direction, we generalize the perspective by studying X -ranks and border ranks of points on varieties of the form $X = \text{Seg}(\mathbb{P}A \times Y) \subset \mathbb{P}(A \otimes W)$, where $Y \subset \mathbb{P}W$ is a projective variety, via a generalized notion of rank and border rank for linear subspaces of W . The special cases where Y is a Veronese variety or Grassmannian arise in applications (see e.g., [23, 7]).

Comparing rank and border rank has been a topic of recent interest, see for example [8, 7, 9, 6, 2, 20, 13, 12].

1.1. Definitions. Let A_1, \dots, A_k be complex vector spaces and let $\text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k) \subset \mathbb{P}(A_1 \otimes \dots \otimes A_k)$ denote the Segre variety of decomposable tensors in $A_1 \otimes \dots \otimes A_k$. Given a tensor $p \in A_1 \otimes \dots \otimes A_k$, define the *rank* (resp. *border rank*) $R_{\text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k)}([p])$ (resp. $\underline{R}_{\text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k)}([p])$) of p to be the smallest r such that there exist

$$[a_1^1 \otimes \dots \otimes a_k^1], \dots, [a_1^r \otimes \dots \otimes a_k^r] \in \text{Seg}(\mathbb{P}A_1 \times \dots \times \mathbb{P}A_k)$$

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with $a_j^i \in A_j$ such that $p = a_1^1 \otimes \cdots \otimes a_k^1 + \cdots + a_1^r \otimes \cdots \otimes a_k^r$ (resp. that there exist curves $a_j^i(t) \subset A_j$ such that $[p] = \lim_{t \rightarrow 0} [a_1^1(t) \otimes \cdots \otimes a_k^1(t) + \cdots + a_1^r(t) \otimes \cdots \otimes a_k^r(t)]$). Sometimes this is called the *tensor (border) rank*. These definitions agree with the definitions in the tensor literature.

More generally, for a projective variety $X \subset \mathbb{P}V$, the X -rank of $[p] \in \mathbb{P}V$, $R_X([p])$, is defined to be the smallest r such that there exist $x_1, \dots, x_r \in X$ such that p is in the span of x_1, \dots, x_r and the X -border rank $\underline{R}_X([p])$ is defined similarly.

We generalize the notion of the X -rank and border rank of a point to a linear subspace:

Let $V = \mathbb{C}^n$, let $\mathbb{P}V$ denote the associated projective space and let $G(k, V)$ denote the Grassmannian of k -planes through the origin in V (equivalently, the set of \mathbb{P}^{k-1} 's in $\mathbb{P}V$). For a subset $Z \subset \mathbb{P}V$, let $\langle Z \rangle \subseteq V$ denote its linear span.

Definition 1.1. For a variety $X \subset \mathbb{P}V$, and $E \in G(k, V)$, define $R_X(E)$, the X -rank of E to be the smallest r , such that there exists $x_1, \dots, x_r \in X$ and $E \subset \langle x_1, \dots, x_r \rangle$. Define $\sigma_{r,k}^0(X) \subset G(k, W)$ to be the set of k -planes of X -rank at most r , and let $\sigma_{r,k}(X) \subset G(k, W)$ denote its Zariski closure. When $k = 1$ we write $\sigma_{r,1}(X) = \sigma_r(X) \subset \mathbb{P}V$ for the r -th secant variety of X . Our notation is such that $\sigma_1(X) = X$.

1.2. Overview. In §2 we revisit two standard results on ranks of tensors to generalize and strengthen them (Theorems 2.1 and 2.4). In §3 we establish basic properties about the varieties $\sigma_{r,k}(X)$ and X -ranks of linear spaces. We briefly mention in §4 a few cases where ranks and border ranks of tensors agree. Some of our results rely on computer calculations, and in §5 we explain the algorithm we used to compute ranks with a computer. In §6 we revisit work of Ja'Ja' [12] on ranks of tensors in $\mathbb{C}^2 \otimes \mathbb{C}^b \otimes \mathbb{C}^c$. When $\mathbf{b} \leq 3$ we describe normal forms for such tensors, explicitly determine their ranks, and give geometric interpretations for the points of a given normal form. In §7 we apply earlier results to the case of partially symmetric tensors, illustrating the utility of the definition of the X -rank of a linear space. We conclude, in §8 by describing normal forms for points in the third secant variety of a Segre variety, and determining or bounding the ranks of such points.

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2. GENERALIZATIONS OF STANDARD RESULTS ON RANKS OF SEGRE PRODUCTS

Let $Y \subset \mathbb{P}W$ be a variety and let $X = \text{Seg}(\mathbb{P}A \times Y) \subset \mathbb{P}(A \otimes W)$ be the Segre product of Y with the projective space $\mathbb{P}A$.

Theorem 2.1. *Let $A' \subset A$ be a linear subspace and let $p \in \mathbb{P}(A' \otimes W)$. Then any expression $p = [v_1 + \cdots + v_s]$ such that some $[v_j] \notin X \cap \mathbb{P}(A' \otimes W)$ has $s > R_X(p)$.*

For any expression $p = \lim_{t \rightarrow 0} [v_1(t) + \cdots + v_s(t)]$ with $v_j(t) \in X$ there exist $w_1(t), \dots, w_s(t)$, such that $p = \lim_{t \rightarrow 0} [w_1(t) + \cdots + w_s(t)]$ with $w_j(t) \in X \cap \mathbb{P}(A' \otimes W)$.

In particular $R_X|_{\mathbb{P}(A' \otimes W)} = R_{X \cap \mathbb{P}(A' \otimes W)}$ and $\underline{R}_X|_{\mathbb{P}(A' \otimes W)} = \underline{R}_{X \cap \mathbb{P}(A' \otimes W)}$.

Theorem 2.1 recovers and strengthens (the ‘‘moreover’’ statement below) the following standard fact (e.g., [4, Prop. 14.35], [9, Prop. 3.1]):

Corollary 2.2. *Let $n > 2$. Let $T \in A_1 \otimes \cdots \otimes A_n$ have rank r . Say $T \in A'_1 \otimes \cdots \otimes A'_n$, where $A'_j \subseteq A_j$, with at least one inclusion proper. Then any expression $T = \sum_{i=1}^{\rho} u_i^1 \otimes \cdots \otimes u_i^n$ with some $u_j^s \notin A'_s$ has $\rho > r$. In particular $R_{\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)}(T) = R_{\text{Seg}(\mathbb{P}A'_1 \times \cdots \times \mathbb{P}A'_n)}(T)$. Moreover $\underline{R}_{\text{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)}(T) = \underline{R}_{\text{Seg}(\mathbb{P}A'_1 \times \cdots \times \mathbb{P}A'_n)}(T)$.*

Proof of Theorem 2.1. Choose a complement $A'' \subset A$ to A' so that $A = A' \oplus A''$.

To prove the assertion regarding rank, write $p = v_1 + \cdots + v_s$, where $v_j = a_j \otimes y_j$ with $y_j \in \hat{Y}$. Write $a_j = b_j + c_j$ with $b_j \in A'$ and $c_j \in A''$. Since $p \in A' \otimes W$, we have $\sum c_j \otimes y_j = 0$. Let $\{e_t\}$ be a basis of A'' . We have $c_j = \xi_j^t e_t$ so $\sum_j \xi_j^t y_j = 0$ for all t . Say e.g., $\xi_s^1 \neq 0$, then we can write y_s as a linear combination of y_1, \dots, y_{s-1} and obtain an expression of rank $s - 1$.

To prove the border rank assertion, for each $v_j(t) = a_j(t) \otimes y_j(t)$ write $a_j(t) = b_j(t) + c_j(t)$ with $b_j(t) \in A'$ and $c_j(t) \in A''$. Also write $p(t) := \sum v_j(t)$, so that $[p] = \lim_{t \rightarrow 0} [p(t)]$. Since $[p] \in \mathbb{P}(A' \otimes W)$,

$$[p] = \lim_{t \rightarrow 0} \left[\sum b_j(t) \otimes y_j(t) \right],$$

hence the claim holds. \square

The following bound on rank (in the three factor case) appears in [4, Prop. 14.45] as an inequality, where it is said to be classical, and as an equality in [10, Thm 2.4]. It is also used e.g., in [12]:

Proposition 2.3. *Let $\phi \in A_1 \otimes \cdots \otimes A_n$, Then $R(\phi)$ equals the number of points of $\text{Seg}(\mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n)$ needed to span $\phi(A_1^*) \subset A_2 \otimes \cdots \otimes A_n$ (and similarly for the permuted statements).*

Here is a generalization and strengthening (the border rank assertion) of Proposition 2.3:

Theorem 2.4. *Let $Y \subset \mathbb{P}W$ and let $X := \text{Seg}(\mathbb{P}A \times Y)$. Given $p \in A \otimes W$, $R_X([p]) = R_Y(p(A^*))$ and $\underline{R}_X([p]) = \underline{R}_Y(p(A^*))$, where on the right hand sides of the equations we have interpreted p as a linear map: $p : A^* \rightarrow W$.*

Proof. We prove the border rank statement, the rank statement is the special case where each curve is constant.

To see $\underline{R}_X([p]) \geq \underline{R}_Y(p(A^*))$, assume $\underline{R}_X([p]) = r$ and write

$$p(t) = a_1(t) \otimes y_1(t) + \cdots + a_r(t) \otimes y_r(t)$$

where $a_i(t) \in A$ and $y_i(t) \in \hat{Y}$ and

$$[p] = \lim_{t \rightarrow 0} [p(t)].$$

If $A' \subset A$ is such that $[p] \in \mathbb{P}(A' \otimes W)$, then by Theorem 2.1 we may assume

$$p(t) \in A' \otimes W \text{ and } a_i(t) \in A'.$$

Replacing A by a smaller vector space if necessary, we may assume $p(t) : A^* \rightarrow W$ is injective for all values of t sufficiently close to 0. Thus the image of $p(t) : A^* \rightarrow W$ determines a curve in the appropriate Grassmannian and the limiting point is $p(A^*)$. Since $[p] = \lim_{t \rightarrow 0} [p(t)]$ as a linear map defined up to scale, $p(A^*) \subset \lim_{t \rightarrow 0} \langle y_1(t), \dots, y_r(t) \rangle$, where the limit is taken in $G(\dim A, W)$. Thus $R_Y(p(t)(A^*)) \leq r$ and $\underline{R}_Y([p]) \leq r$.

To see $\underline{R}_X([p]) \leq \underline{R}_Y(p(A^*))$, assume $\underline{R}_Y(p(A^*)) = r$ and that there exist curves $y_j(t) \subset \hat{Y}$, such that $p(A^*) \subset \lim_{t \rightarrow 0} \langle y_1(t), \dots, y_r(t) \rangle$, where again we may assume the dimension of the span of the $y_s(t)$ is constant for $t \neq 0$ and we are taking the limit in the appropriate Grassmannian. Let a^1, \dots, a^a be a basis of A^* . Write $p(a^j) = \lim_{t \rightarrow 0} \sum_s c^{js}(t) y_s(t)$ for some functions $c^{js}(t)$, so $p = \sum_j a_j \otimes [\lim_{t \rightarrow 0} \sum_s c^{js}(t) y_s(t)]$. Consider the curve

$$\begin{aligned} p(t) &= \sum_{j,s} a_j \otimes c^{js}(t) y_s(t) \\ &= \sum_s \left[\sum_j c^{js}(t) a_j \right] \otimes y_s(t). \end{aligned}$$

Thus $R_X([p(t)]) \leq r$ for $t \neq 0$ and since $\lim_{t \rightarrow 0} [p(t)] = [p]$, the claim $\underline{R}_X([p]) \leq r$ follows. \square

Example 2.5. Let $Y = v_3(\mathbb{P}^1)$ and $X = \text{Seg}(\mathbb{P}A \times v_3(\mathbb{P}^1))$. Let $p = a_1 \otimes x^3 + a_2 \otimes x^2 y$. Then $R_X(p) = R_Y(p(A^*)) = 3$ and $\mathbf{R}_X(p) = \mathbf{R}_Y(p(A^*)) = 2$. To see $\mathbf{R}_Y(p(A^*)) = 2$, note that

$$p(A^*) = \lim_{t \rightarrow 0} \langle x^3, (x + ty)^3 \rangle.$$

To see $\mathbf{R}_X(p) = 2$, consider the curve

$$p(t) = \frac{1}{t} [(a_2 + ta_1) \otimes (x + ty)^3 - a_2 \otimes x^3].$$

Corollary 2.6. *The maximum rank of an element of $A_1 \otimes \cdots \otimes A_n$, where*

$$\dim A_1 \leq \cdots \leq \dim A_n,$$

is at most $\dim A_1 \cdots \dim A_{n-1}$. In particular, the maximum rank of an element of $(\mathbb{C}^m)^{\otimes n}$ is at most m^{n-1} , compared with the maximum border rank, which is usually $\frac{m^n}{(m-1)n+1} \sim \frac{m^{n-1}}{n}$.

Note that this upper bound is much smaller than the bound in [20, Prop. 5.1].

3. GENERAL FACTS ABOUT RANK AND BORDER RANK OF LINEAR SPACES

The following facts are immediate consequences of the definitions of $R_X(E)$, $\mathbf{R}_X(E)$:

Proposition 3.1. *Let $X \subset \mathbb{P}V = \mathbb{P}^N$ be a variety of dimension n not contained in a hyperplane, let $E \in G(k, V)$. Then*

- (i) $R_X(E) \geq \mathbf{R}_X(E) \geq k$ and $\sigma_{r_1, k}(X) \subset \sigma_{r_2, k}(X)$ whenever $r_1 \leq r_2$.
- (ii) If X is irreducible and $E \in \sigma_{r, k}(X)$ is a general point, then $R_X(E) = \mathbf{R}_X(E)$.
- (iii) $R_X(E) = k$ iff $\langle X \cap \mathbb{P}E \rangle = \mathbb{P}E$. In particular, if $\text{codim} X \leq k$ and E intersects X transversally, then $R_X(E) = k$ and thus $\sigma_{k, k}(X) = G(k, N+1)$.
- (iv) $\sigma_{r, N-n}(X) = G(r, N+1)$.
- (v) For $r \leq N+1$, $R_X(E) \leq r$ if and only if $\exists F \in \sigma_{r, r}^0(X)$ such that $E \subset F$ and similarly $\mathbf{R}_X(E) \leq r$ if and only if $\exists F \in \sigma_{r, r}(X)$ such that $E \subset F$.

Proposition 3.2. *Let $X \subset \mathbb{P}V$ be a variety of dimension n . If X is not contained in a hyperplane, then $\dim \sigma_{r, r}(X) = \min\{rn, \dim G(r, V)\}$. The expected dimension of $\sigma_{r, s}(X)$ is $\min\{rn + s(r-s), \dim G(s, V)\}$.*

Proof. If $n+r > \dim V$, then a general $E \in G(r, V)$ is spanned by points of X and thus

$$\sigma_{r, r}(X) = G(r, V).$$

Otherwise, if $n+r \leq \dim V$, then a general collection of $r+n$ points $x_1, \dots, x_r, y_1, \dots, y_n$ on \hat{X} are linearly independent. Let $y_i(t)$ now be curves in \hat{X} such that all the y_i to converge to x_1 in such a way that, the limit of the span of x_1, y_1, \dots, y_n is the affine tangent space of X at $[x_1]$. We see that if our points are chosen generally $\hat{T}_{[x_1]}X$ will intersect $\langle x_2, \dots, x_r \rangle$ only at the origin.

Thus we can choose r points $x_1, \dots, x_r \in \hat{X}$ such that the points are in general linear position and also

$$\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r \rangle \cap \hat{T}_{[x_i]}X = 0$$

for any i .

Consider $[x_1 \wedge \cdots \wedge x_r] \in \sigma_{r, r}(X)$. Take a curve $x_1(t) \wedge \cdots \wedge x_r(t)$ with $x_j(0) = x_j$, so $x_j(0)' \in \hat{T}_{[x_j]}X$. Differentiate at $t=0$ to obtain:

$$\begin{aligned} \hat{T}_{[x_1 \wedge \cdots \wedge x_r]} \sigma_{r, r}(X) = \\ (\hat{T}_{[x_1]}X) \wedge x_2 \wedge \cdots \wedge x_r + x_1 \wedge (\hat{T}_{[x_2]}X) \wedge x_3 \wedge \cdots \wedge x_r + \cdots + x_1 \wedge \cdots \wedge x_{r-1} \wedge (\hat{T}_{[x_r]}X). \end{aligned}$$

Mod-ing out by $x_1 \wedge \cdots \wedge x_r$ we obtain a direct sum:

$$(\hat{T}_{[x_1]}X) \wedge x_2 \wedge \cdots \wedge x_r \oplus x_1 \wedge (\hat{T}_{[x_2]}X) \wedge x_3 \wedge \cdots \wedge x_r \oplus \cdots \oplus x_1 \wedge \cdots \wedge x_{r-1} \wedge (\hat{T}_{[x_r]}X) \bmod x_1 \wedge \cdots \wedge x_r.$$

This holds because say e.g., $v \in x_1 \wedge \cdots \wedge x_{r-1} \wedge (\hat{T}_{[x_r]}X)$ were in the span of the other summands. Each of the other summands is of the form $(stuff) \wedge x_r$, so v must be of this form as well, but $x_r \notin \langle x_1, \dots, x_{r-1} \rangle$. Thus, if X is not contained in a linear space, $\dim \sigma_{r,r}(X) = rn$.

We may realize $\sigma_{r,s}(X)$ as the *collapsing* of a Grassmann bundle over $\sigma_{r,r}(X)$ in the sense of [24]. Namely let $\mathcal{S} \rightarrow G(r, V)$ denote the tautological rank r subspace bundle and let $G(s, \mathcal{S}) \rightarrow G(r, V)$ the corresponding Grassmann bundle. There is a natural map from the total space of $G(s, \mathcal{S})$ onto $G(s, V)$ and by Proposition 3.1(v), $\sigma_{r,s}(X)$ is the image of $G(s, \mathcal{S})|_{\sigma_{r,r}(X)}$. Thus if this map is generically finite to one, $\dim \sigma_{r,s}(X) = \dim \sigma_{r,r}(X) + s(r - s)$. \square

Corollary 3.3. *Let A, W be two vector spaces with $\dim A \leq \dim W$. Consider the rational map $\pi : \mathbb{P}(A \otimes W) \dashrightarrow G(\dim A, W)$, with $\pi(p) := p(A^*)$ whenever $p(A^*)$ has maximal dimension. Then:*

- (i) π is the quotient of $\mathbb{P}(A \otimes W)$ by $PGL(A)$.
- (ii) π is $PGL(W)$ equivariant.
- (iii) for $r \geq \dim A$ and any subvariety $Y \subset \mathbb{P}W$,

$$\sigma_r(\mathbb{P}A \times Y) = \overline{\pi^{-1}(\sigma_{r, \dim A}(Y))}.$$

Corollary 3.4. *If $r \leq \dim A \leq \dim W$ and $Y \subset \mathbb{P}W$ is not contained in any hyperplane, then $\sigma_r(\text{Seg}(\mathbb{P}A \times Y))$ is of the expected dimension $r(\dim A + \dim Y) - 1$.*

Proof. If $r = \dim A$, then by Corollary 3.3(i) and (iii),

$$\dim(\sigma_r(\mathbb{P}A \times Y)) = \dim(\sigma_{r,r}(Y)) + \dim PGL(A).$$

By Proposition 3.2:

$$\dim(\sigma_r(\mathbb{P}A \times Y)) = r \dim Y + r^2 - 1 = r(r + \dim Y) - 1$$

as claimed.

If $r < \dim A$, then $\sigma_r(\mathbb{P}A \times Y)$ is swept out by smaller secant varieties:

$$\sigma_r(\text{Seg}(\mathbb{P}A \times Y)) = \bigcup_{A' \subset A, \dim A' = r} \sigma_r(\text{Seg}(\mathbb{P}A' \times Y)).$$

Since Y is nondegenerate and $\dim W > r$, it follows, that for a general point $[p] \in \sigma_r(\text{Seg}(\mathbb{P}A \times Y))$ there is a unique linear subspace $A' \subset A$ of dimension r such that $[p] \in \mathbb{P}(A' \otimes V)$. Thus:

$$\begin{aligned} \dim(\sigma_r(\text{Seg}(\mathbb{P}A \times Y))) &= \dim G(r, A) + \dim(\sigma_r(\mathbb{P}A' \times Y)) \\ &= r(\dim A - r) + r(r + \dim Y) - 1 \\ &= r(\dim A + \dim Y) - 1. \end{aligned}$$

\square

In [2], scheme-theoretic methods are used for studying rank. For some varieties X , every point on $\sigma_r(X)$ is contained in the linear span of a degree r subscheme of X (see [2, Prop. 2.8]). For $X \subset \mathbb{P}V$ consider the irreducible component H_r of the Hilbert scheme $\text{Hilb}(X)$ containing schemes, which are r distinct points with reduced structure. Consider the rational map

$$\varphi : H_r \dashrightarrow G(r, V),$$

which sends a subscheme $Z \subset X$ to its scheme-theoretic linear span.

Proposition 3.5. *If φ is a regular map, i.e., if each $Z \in H_r$ imposes independent conditions to linear forms, then for every $E \in \sigma_{r,k}(X)$ there exists a 0-dimensional scheme $Z \subset X$ of degree r such that E is contained in the scheme-theoretic span of Z .*

Thus the methods used in [2] may be used to study $Seg(\mathbb{P}A \times v_d(\mathbb{P}^n))$ and its secant varieties.

4. SITUATIONS WHERE RANKS AND BORDER RANKS OF POINTS COINCIDE

Proposition 4.1. *Let $X \subset \mathbb{P}V$ be one of $v_2(\mathbb{P}^n)$ (the rank one symmetric $(n+1) \times (n+1)$ matrices), $G(2,n)$ (the rank two skew-symmetric $n \times n$ matrices), $Seg(\mathbb{P}^a \times \mathbb{P}^b)$, the Cayley plane $\mathbb{O}\mathbb{P}^2$, or the 10-dimensional spinor variety \mathbb{S}_5 , (i.e, X is a sub-cominuscule variety, see [17]). Then for all $p \in \mathbb{P}V$, $R_X(p) = \underline{\mathbf{R}}_X(p)$.*

Proof. In all these cases the only orbits are the successive secant varieties, see, e.g., [18]. \square

A consequence of [5, Thm 2.4.2], which states that for r in the range of the hypotheses in Proposition 4.2, $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) = \sigma_r(Seg(\mathbb{P}(A_1 \otimes \cdots \otimes \mathbb{P}A_{n-1}) \times \mathbb{P}A_n))$, and Proposition 4.1 is:

Proposition 4.2. *Consider $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ where $\dim A_s = \mathbf{a}_s$, $1 \leq s \leq n$. If $\mathbf{a}_n > r \geq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$, then $\underline{\mathbf{R}}_X(p) = r$ implies $R_X(p) = r$.*

5. COMPUTER ALGORITHM TO CALCULATE RANK

Consider a subvariety $X \subset \mathbb{P}V$ and a linear subspace $E \subset V$. The algorithm described in this section searches for all linear spaces $W \subset V$ of a fixed dimension, which contain E and are spanned by points of X . We are interested in determining when the set of solutions is empty or non-empty to get lower or upper bounds on rank. The case we are mainly interested in is when $X = Seg(\mathbb{P}B \times \mathbb{P}C)$. Here are several examples of ranks we calculated using the algorithm:

Proposition 5.1. (i) *In $\mathbb{C}^4 \otimes \mathbb{C}^4$ consider the two dimensional subspace:*

$$E := \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

Then $R_{Seg(\mathbb{P}^3 \times \mathbb{P}^3)}(E) = 6$.

(ii) *In $\mathbb{C}^6 \otimes \mathbb{C}^6$ consider the two dimensional subspace:*

$$E := \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

Then $R_{Seg(\mathbb{P}^5 \times \mathbb{P}^5)}(E) = 9$.

(iii) *In $\mathbb{C}^3 \otimes \mathbb{C}^3$ consider the three dimensional subspace:*

$$E := \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

Then $R_{Seg(\mathbb{P}^2 \times \mathbb{P}^2)}(E) = 6$.

Given a linear subspace $W \subset V$, to verify if W is spanned by points of X it is enough to perform the following calculation: Consider the embedding map $\phi : \mathbb{P}W \rightarrow \mathbb{P}V$ and let $Z = \phi^{-1}(X)$. Now W is spanned by points of X if and only if the radical ideal of $Z \subset \mathbb{P}W$ has no linear terms. The calculation of the radical might be very expensive, but we are only interested in the linear part of the radical, we explain below how to simplify the calculation with this in mind.

Fix $E \subset V$, a positive integer k , and let $r := \dim E + k$. We are looking for W of dimension r . We want to find the set of collections of k vectors x_i in \hat{X} , such that $W = \langle E, x_1, \dots, x_k \rangle$ is spanned by points of X . Give \mathbb{C}^k linear coordinates (f_1, \dots, f_k) and consider the rational map

$$\begin{aligned} (\hat{X})^k \times E \times \mathbb{C}^k &\dashrightarrow \mathbb{P}V \\ (x_1, \dots, x_k, e, (f_1, \dots, f_k)) &\mapsto [e + f_1x_1 + \dots + f_kx_k] \end{aligned}$$

which descends to a rational map:

$$\phi : (\hat{X})^k \times \mathbb{P}(E \times \mathbb{C}^k) \dashrightarrow \mathbb{P}V.$$

For fixed x_i 's the image is just the linear subspace spanned by E and the x_i 's, and is our candidate for W .

As in the previous case, it is “enough” to take $Z := \phi^{-1}(X)$ with reduced structure and take the equations which are linear with respect to the variables e and x_i . And as above, if W is spanned by points of X , then the linear equations vanish for some parameters x_i . So we must consider the locus in $(\hat{X})^k$ where all the coefficients of the linear equations of Z vanish. If the locus is empty, then E has rank at least $r + 1$. If it is not empty, then we have candidates for the generators of W . These are not yet necessarily the solutions, because the scheme Z' defined by the vanishing of these coefficients and reduced equations of Z is not necessarily reduced. So we must then find the linear span of reduced subscheme of Z' and so on.

In the cases we considered, particularly in case (ii) from Proposition 5.1, it was easier to calculate the set-theoretic linear span of a subscheme using even more steps. Namely, before finding the radical of the ideal of Z , already half way through one might encounter a linear equation. Once one has this equation, the coefficients of the linear equation are put into the ideal and one keeps calculating the radical of the new ideal, watching out for more linear equations.

The affine cone \hat{X} can be replaced with another variety which parametrizes \hat{X} . For instance, when $X = \text{Seg}(\mathbb{P}B \times \mathbb{P}C)$, we replace \hat{X} with $B \times C$, which is more convenient for calculations. When $X = v_d(\mathbb{P}B)$, we can parametrize \hat{X} with B .

6. RANKS OF POINTS IN $\mathbb{C}^2 \otimes \mathbb{C}^b \otimes \mathbb{C}^c$

6.1. Kronecker's normal form. Kronecker determined a normal form for pencils of matrices, i.e., two dimensional linear subspaces of $B \otimes C$ up to the action of $GL(B) \times GL(C)$. It is convenient to use matrix notation so choose bases of B, C and write the family as $sX + tY$, where $X, Y \in B \otimes C$ and $s, t \in \mathbb{C}$. (Kronecker's classification works over arbitrary fields, as does J. Ja'Ja's classification of rank, but we only present the results over \mathbb{C} .) The result is as follows (see, e.g., [11, Chap. XII]):

Define the $\epsilon \times (\epsilon + 1)$ matrix

$$L_\epsilon = L_\epsilon(s, t) = \begin{pmatrix} s & t & & & \\ & \ddots & \ddots & & \\ & & & s & t \end{pmatrix}.$$

It only remains to change basis in B by sending b_1 to $\frac{1}{\mu_2}b_1$ and b_2 to $\frac{1}{\nu_2}b_2$ to show the pencil is equivalent to:

$$s_3 \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} + t_3 \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Thus every two such pencils are equivalent.

If F is 4×4 it is no longer possible to normalize away all constants in the case

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \mu & 1 \\ & & & \mu \end{pmatrix}.$$

Essentially because of this, the only spaces of tensors $\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$, $2 \leq \mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$, that admit normal forms, i.e., have a finite number of $GL_{\mathbf{a}} \times GL_{\mathbf{b}} \times GL_{\mathbf{c}}$ -orbits, are $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{c}}$ and $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}}$ (see [14, 15]). Moreover, in these cases, any tensor lies in a $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ in the first case and a $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^6$ in the second.

Theorem 6.1 (Ja'Ja'). [12] *A pencil of the form (6.1) has rank $\sum \epsilon_i + \sum \mu_j + f + q + p + M$ where $0 \leq M \leq \mu$, where μ is the maximum number of nontrivial Jordan chains associated with a given eigenvalue of F . If $\mu > 0$, then $M \geq 1$.*

The idea of the proof is as follows: the normalized blocks are treatable by a direct calculation, the only subtle part is to show that one can not have a lower rank for the sum of two blocks than the sum of the ranks of each block. The $f \times f$ pencil $sId + tF$ has rank f iff F can be diagonalized. Most matrices are diagonalizable, so if F is not, a small perturbation of it will be diagonalizable. Ja'Ja' shows that each Jordan block can be perturbed to be diagonalizable by a rank one matrix. In fact if one selects a rank one matrix at random, with probability one, the perturbed Jordan block will be diagonalizable. Ja'Ja' gives a convenient such matrix using rational canonical form instead of Jordan canonical form. The inconclusiveness of the result on ranks is because he did not determine if one could perturb by using a higher rank matrix that made more than one chain associated to given eigenvalue diagonalizable at once.

Taking into account the various cases Theorem 6.1 implies:

Corollary 6.2.

- *The maximum rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{c}}$ is 3 if $\mathbf{c} = 2, 3$ and 4 otherwise.*
- *The maximum rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}}$ is 3 or 4 if $\mathbf{c} = 3$, is \mathbf{c} if $\mathbf{c} = 4, 5, 6$ and 6 otherwise.*
- *The maximum rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ is 7.*

Theorem 6.1 also implies that there are no tensors in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$ respectively of ranks 7, 10 when \mathbf{b} is respectively 4, 6.

Theorem 6.3. *The maximum rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ is 6, and in $\mathbb{C}^2 \otimes \mathbb{C}^6 \otimes \mathbb{C}^6$ is 9.*

The ranks 6 and 9 respectively are obtained for the examples in Proposition 5.1.

In general, Theorem 6.1 says there is no tensor in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$ of rank $\lfloor \frac{3\mathbf{b}}{2} \rfloor + 1$, but guarantees the existence, e.g., of tensors of rank $\frac{4\mathbf{b}}{3}$ when \mathbf{b} is a multiple of 3.

6.2. Orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{c}}$. Here and in what follows $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. If $\mathbf{c} = 3$ then $\sigma_3(X) = \mathbb{P}(A \otimes B \otimes C)$. In the table below we list the representatives of all the orbits of action of $GL(A) \times GL(B) \times GL(C)$ on $\mathbb{P}(A \otimes B \otimes C)$.

orbit closure	dimension	Kronecker normal form	$\underline{\mathbf{R}}$	R
X	$\mathbf{c} + 1$	$a_1 \otimes b_1 \otimes c_1$	1	1
Sub_{221}	$\mathbf{c} + 3$	$a_2 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_1$	2	2
Sub_{122}	$2\mathbf{c} + 1$	$a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1$	2	2
Sub_{212}	$2\mathbf{c} + 1$	$a_1 \otimes b_1 \otimes c_2 + a_2 \otimes b_1 \otimes c_1$	2	2
$\tau(X)$	$2\mathbf{c} + 2$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes b_1 \otimes c_2$	2	3
$\sigma_2(X) = Sub_{222}$	$2\mathbf{c} + 3$	$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2$	2	2
X_*^\vee	$3\mathbf{c} + 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes b_2 \otimes c_1$	3	3
$\sigma_3(X)$	$3\mathbf{c} + 2$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3)$	3	3
$\mathbb{P}(A \otimes B \otimes C)$	$4\mathbf{c} - 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	4	4

The first and third are not pencils of matrices. In terms of matrices, the other cases are

$$(s, t), \begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s & t \\ & s \end{pmatrix}, \begin{pmatrix} s & & \\ & t & \\ & & s \end{pmatrix}, \begin{pmatrix} s & t & \\ & s & t \\ & & s \end{pmatrix}, \begin{pmatrix} s & t & & \\ & s & t & \\ & & s & t \\ & & & s \end{pmatrix},$$

Geometric explanations: $Sub_{ijk} \subset \mathbb{P}(A \otimes B \otimes C)$ is the set of tensors $[p] \in \mathbb{P}A \otimes B \otimes C$ such that there exists $A' \subset A$, $B' \subset B$, $C' \subset C$ respectively of dimensions i, j, k such that $p \in A' \otimes B' \otimes C'$. In other words, Sub_{ijk} is the projectivization of the image of the vector bundle $\mathcal{S}_{G(i,A)} \otimes \mathcal{S}_{G(j,B)} \otimes \mathcal{S}_{G(k,C)} \rightarrow G(i,A) \times G(j,B) \times G(k,C)$ in $A \otimes B \otimes C$, where $\mathcal{S}_{G(k,V)} \rightarrow G(k,V)$ is the tautological vector bundle whose fiber over the point E is the linear space E . $\tau(X)$ is the tangential variety to the Segre variety, $X_* \subset \mathbb{P}(A^* \otimes B^* \otimes C^*)$ is the Segre variety in the dual projective space and $X_*^\vee \subset \mathbb{P}(A \otimes B \otimes C)$ is its dual variety. The point of $\tau(X)$ is tangent to the point $a_1 \otimes b_2 \otimes c_1$, the point of X_*^\vee contains the tangent plane to the $(\mathbf{c} - 3)$ -parameter family of points $a_2^* \otimes b_1^* \otimes (s_2 c_2^* + s_2 c_4^* + s_2 c_5^* \cdots + s_c c_c^*)$, where (a_j^*) is the dual basis to (a_j) of A etc.. The dual variety $(X_*)^\vee$ is degenerate (i.e., not a hypersurface) except when $\mathbf{c} \leq 3$.

Note $\sigma_3(X) = \sigma_3(\mathbb{P}(A \otimes B) \times \mathbb{P}C)$ which causes it to be degenerate with defect three.

All these orbits but the last are inherited from the $\mathbf{c} = 3$ case. The last is inherited from the $\mathbf{c} = 4$ case.

6.3. $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. For future use, let $\mathbf{c} = 3$. First, we inherit all the orbits from the $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ case, i.e., all the orbit closures above except for the last one. They become subvarieties of Sub_{223} with the same normal forms, ranks and border ranks - the new dimensions are as follows:

orbit	dimension
$X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$	$\mathbf{c} + 2$
Sub_{221}	$\mathbf{c} + 4$
Sub_{212}	$\mathbf{c} + 4$
Sub_{122}	$2\mathbf{c}$
$\tau(X)$	$2\mathbf{c} + 4$
$Sub_{222} = \sigma_2(X)$	$2\mathbf{c} + 5$
$Seg_*^\vee \subset Sub_{223}$	$3\mathbf{c} + 4$
Sub_{223}	$4\mathbf{c} + 1$

The new orbit closures are as follows:

orbit	dimension	Kronecker normal form	\mathbf{R}	R
Sub_{133}	$3\mathbf{c} - 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3)$	3	3
$Seg_*^\vee \subset Sub_{232}$	$2\mathbf{c} + 6$	$a_1 \otimes (b_1 \otimes c_1 + b_3 \otimes c_2) + a_2 \otimes b_2 \otimes c_1$	3	3
Sub_{232}	$2\mathbf{c} + 7$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_2 \otimes c_1 + b_3 \otimes c_2)$	3	3
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_3 \otimes c_3)$	3	4
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes b_3 \otimes c_3$	3	3
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes b_2 \otimes c_1$	3	4
X_*^\vee	$3\mathbf{c} + 7$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes (b_2 \otimes c_1 + b_3 \otimes c_2)$	3	4
$\mathbb{P}(A \otimes B \otimes C)$	$3\mathbf{c} + 8$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes (b_2 \otimes c_2 - b_3 \otimes c_3)$	3	3

The associated $f \times f$ matrices, when 3×3 , are given by (6.2). Note that the second and third cases are not really new. The unnamed orbits are various components of the singular locus of X_*^\vee , see [15] for descriptions.

$Seg_*^\vee \subset Sub_{223}$ is the subvariety of Sub_{223} , obtained from the sub-fiber bundle of $\mathcal{S}_{G(2,A)} \otimes \mathcal{S}_{G(2,B)} \otimes \mathcal{S}_{G(3,C)}$, whose fiber in $A' \otimes B' \otimes C'$ (where $\dim A' = \dim B' = 2$, $\dim C' = 3$) is $\hat{S}eg(\mathbb{P}A'^* \times \mathbb{P}B'^* \times \mathbb{P}C'^*)^\vee \subset A' \otimes B' \otimes C'$.

6.4. $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$. We have all the orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, which are all contained in Sub_{233} and thus should be re-labeled as such, the last one becomes Sub_{233} , their dimensions are given in the above charts by setting $\mathbf{c} = 4$. The new orbit closures are:

orbit	dimension	normal form	\mathbf{R}	R
Sub_{224}	$4\mathbf{c} + 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	4	4
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_4) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3)$	4	4
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_4) + a_2 \otimes b_1 \otimes c_2$	4	4
		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_4) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	4	4
X_*^\vee	$10\mathbf{c} - 18$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_4) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3)$	4	4
$\mathbb{P}(A \otimes B \otimes C)$	$6\mathbf{c} - 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3 + b_3 \otimes c_4)$	4	4

The unlabeled orbit closures are various components of $(X_*^\vee)_{sing}$.

6.5. $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^c$. We have all the orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$, which are all contained in Sub_{234} and thus should be re-labeled as such, the last one becomes Sub_{234} , plus:

orbit	dimension	normal form	\mathbf{R}	R
X_*^\vee	$5\mathbf{c} + 2$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_4 + b_3 \otimes c_5) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	4	5
$Sub_{235} = \sigma_5(X)$	$5\mathbf{c} + 4$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_5) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4 + b_3 \otimes c_5)$	5	5
$\mathbb{P}(A \otimes B \otimes C)$	$6\mathbf{c} - 1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_5 \otimes c_5) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4 + b_3 \otimes c_6)$	6	6

When $\mathbf{c} = 5$, $Sub_{235} = \mathbb{P}(A \otimes B \otimes C)$ and the last normal form does not occur.

7. APPLICATION TO $X = Seg(\mathbb{P}A \times v_2(\mathbb{P}W)) \subset \mathbb{P}(A \otimes S^2W)$

This case is closely related to the determination of ranks of tensors with symmetric matrix slices, an often-studied case in applications, see, e.g., [23] and the references therein. (If the generalized Comon conjecture (see, e.g., [3]) is true here, this case is the same as the case of tensor rank of partially symmetric tensors.)

This case is easier than that of $Seg(\mathbb{P}A \times \mathbb{P}W \times \mathbb{P}W)$ because a symmetric matrix is always diagonalizable.

Subcase a = 2. Assume without loss of generality that $p(A^*) \not\subset S^2W'$ for some linear space $W' \subset W$, and that the image is two-dimensional. If there is an element of maximal rank in $p(A^*)$, then $R_X(p) = \dim W$, because the second matrix must be diagonalizable.

On the other hand, pencils of symmetric matrices of bounded rank can be classified by the Kronecker normal form: just take something of the normal form (6.1) where $p = q$ and $\epsilon_j = \eta_j$, and permute the rows and columns so that it becomes a symmetric matrix. We conclude:

Theorem 7.1. *Let $X = \text{Seg}(\mathbb{P}^1 \times v_2(\mathbb{P}^{n-1})) \subset \mathbb{P}(A \otimes S^2W)$. Let $p \in A \otimes S^2W$, and let $p(A^*)$ have Kronecker normal form with indices g, f , and ϵ_j , so $n = \dim W = \sum 2\epsilon_j + f$. Then $R_X(p) = n + 2g$. In particular, the maximum X -rank of an element of $\mathbb{C}^2 \otimes S^2\mathbb{C}^n$ is $\lfloor \frac{3n}{2} \rfloor$.*

8. RANKS AND NORMAL FORMS IN $\sigma_3(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$

We assume throughout this section that $n \geq 3$.

Recall that for any smooth variety X , if $x \in \sigma_2(X)$, then either $x \in X$, $x \in \sigma_2^0(X)$ or x lies on an embedded tangent line to X .

8.1. $\sigma_2(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$ case.

Theorem 8.1. *Let $X = \text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ be a Segre variety. There is a normal form for points $x \in \hat{\sigma}_2(X)$: $x = x = a_0^1 \otimes \cdots \otimes a_0^n$ for a point of X , $x = a_0^1 \otimes \cdots \otimes a_0^n + a_1^1 \otimes \cdots \otimes a_1^n$ for a point on a secant line to X (where for this normal form, we don't require all the a_1^i to be independent of the a_0^i), and have ranks 1, 2, and for each $J \subseteq [n]$, $|J| > 2$, the normal form*

$$(8.1) \quad x = a_0^1 \otimes \cdots \otimes a_0^n + \sum_{j_i \in J} a_0^1 \otimes \cdots \otimes a_0^{j_i-1} \otimes a_1^{j_i} \otimes a_0^{j_i+1} \otimes \cdots \otimes a_0^n$$

which has rank $|J|$. In particular, all ranks from 1 to n occur for elements of $\sigma_2(X)$.

Proof. All the assertions except for the rank of x in (8.1) are immediate. The rank of x is at most $|J|$ because there are $|J| + 1$ terms in the summation but the first can be absorbed into any of the others (e.g. using $a_0^{j_i} + a_1^{j_i}$ instead of $a_1^{j_i}$).

Assume WLOG $|J| = n$ and work by induction. First say $n = 3$, then Corollary 6.2 establishes this base case.

Now assume we have established the result up to $n - 1$, and consider the $x(A_1^*)$. It is spanned by

$$a_0^2 \otimes \cdots \otimes a_0^n, \sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n.$$

By induction, the second vector has rank $n - 1$. It remains to show that there is no expression of the second vector as a sum of $n - 1$ decomposable tensors where one of terms is a multiple of $a_0^2 \otimes \cdots \otimes a_0^n$. Say there were, where $a_0^2 \otimes \cdots \otimes a_0^n$ appeared with coefficient λ , then the tensor $\sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n - \lambda a_0^2 \otimes \cdots \otimes a_0^n$ would have rank $n - 2$, but setting $\tilde{a}_1^2 = a_1^2 - \lambda a_0^2$ and $\tilde{a}_1^j = a_1^j$ for $j \in \{3, \dots, n\}$, this would imply that

$$\sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes \tilde{a}_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n$$

had rank $n - 2$, a contradiction. \square

8.2. $\sigma_3(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)) \setminus \sigma_2(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$ case. Let

$$x \in \sigma_3(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)) \setminus \sigma_2(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)).$$

The standard types of points are: a point on an honest secant \mathbb{P}^2 (which has rank 3), a point on the plane spanned by a point of the Segre and a tangent \mathbb{P}^1 to the Segre, which has rank at most $n + 1$, and a point of the form $y + y' + y''$ where $y(t)$ is a curve on $\hat{\text{Seg}}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)$.

The latter type of point has rank at most $\binom{n+1}{2}$ because a generic such point is of the form

$$\begin{aligned} & a_0^1 \otimes \cdots \otimes a_0^n + \sum_j a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n \\ & + \sum_{j < k} a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^{k-1} \otimes a_1^k \otimes a_0^{k+1} \otimes \cdots \otimes a_0^n \\ & + \sum_j a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_2^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n \end{aligned}$$

The first term and second set can be folded into the last term, giving the estimate. One can show that the above types of points exhaust the possibilities for $\sigma_3(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$.

When $n = 3$, we showed (see Proposition 5.1(iii)) the rank of the 3 plane given by

$$\begin{pmatrix} c_2 & c_1 & c_0 \\ c_1 & c_0 & 0 \\ c_0 & 0 & 0 \end{pmatrix}$$

is six. Thus:

Proposition 8.1. *The rank of a point of $\sigma_3(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ of the form $y + y' + y''$ is 6. In particular, the maximum rank of a point of $\sigma_3(\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is 6.*

Starting with $\sigma_4(\text{Seg}(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$ there are exceptional limit points and these points turn out to be important - they are used in Schonhage's approximate algorithm to multiply 3×3 matrices using 21 multiplications, see [3].

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