A GEOMETRIC STUDY OF STRASSEN'S ASYMPTOTIC RANK CONJECTURE AND ITS VARIANTS

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Abstract. We establish basic information about the set of *tight* tensors, the tensors with continuous regular symmetry. Our motivation is Strassen's astounding *asymptotic rank conjecture* that the asymptotic rank of any tight tensor is minimal. In particular, we determine the dimension of the set of tight tensors. Surprisingly we prove this dimension equals the dimension of the set of *oblique tensors*, a less restrictive class of tensors that Strassen identified as useful for his laser method.

1. Introduction

This paper is part of a series where we explore new paths towards proving upper complexity bounds for matrix multiplication using geometric methods, and re-examine existing paths from a geometric perspective. Sometimes, in an effort to resolve a conjecture that seems out of reach, one poses an even stronger conjecture, with the hope that the existing conjecture might more easily be resolved in a larger context, or, that disproving the stronger conjecture might be accessible and lead to insights regarding the original conjecture. This paper examines several generalizations of the conjecture that the exponent of matrix multiplication is two. Independent of matrix multiplication, our study sheds light on the geometry of tensors. We begin with some necessary definitions.

Throughout the paper, A, B, C denote complex vector spaces of dimension $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. We fix bases $\{a_0, \dots, a_{\mathbf{a}-1}\}$ of A with dual basis $\{\alpha^0, \dots, \alpha^{\mathbf{a}-1}\}$ of A^* and similarly for B and C. Given two tensors $T_1 \in A_1 \otimes B_1 \otimes C_1$ and $T_2 \in A_2 \otimes B_2 \otimes C_2$, one can regard the tensor $T_1 \otimes T_2$ as an element of $(A_1 \otimes A_2) \otimes (B_1 \otimes B_2) \otimes (C_1 \otimes C_2)$. This is called *Kronecker product* of T_1 and T_2 and it is denoted by $T_1 \boxtimes T_2$. One can define higher products iteratively; for $T \in A \otimes B \otimes C$, let $T^{\boxtimes N} \in (A^{\otimes N}) \otimes (B^{\otimes N}) \otimes (C^{\otimes N})$ denote the N-th Kronecker power of T.

In several disciplines, one is interested in the *asymptotic properties* of a tensor under Kronecker powers, that is properties of T which can be detected studying $T^{\boxtimes N}$ for $N \to \infty$. We refer to [CVZ18, §1.1] and the references therein for a thorough discussion on the applications.

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Motivated by the study of the complexity of matrix multiplication, and influenced by classical information theory, Strassen [Str87, Str91] proposed to study the rate of growth of the complexity of the bilinear map $T^{\boxtimes N} : (A^{\otimes N})^* \times (B^{\otimes N})^* \to C^{\otimes N}$ for a given tensor T. Indeed, the matrix multiplication tensor $M_{\langle \mathbf{n} \rangle} \in \mathrm{Mat}^*_{\mathbf{n}} \otimes \mathrm{Mat}^*_{\mathbf{n}} \otimes \mathrm{Mat}_{\mathbf{n}}$, namely the bilinear map sending two matrices of size $\mathbf{n} \times \mathbf{n}$ to their product, has the remarkable property that $M^{\boxtimes N}_{\langle \mathbf{n} \rangle} = M_{\langle \mathbf{n}^N \rangle}$, so studying its Kronecker powers is equivalent to studying larger matrix multiplication tensors. The asymptotic complexity of a bilinear map, in the sense described above, is controlled by the so-called *asymptotic rank* of the corresponding tensor, see Definition 1.2 below. The following astounding conjecture, not attributed to anyone, is commonly believed true in the theoretical computer science community:

Conjecture 1.1. *The asymptotic rank of $M_{\langle \mathbf{n} \rangle}$ is $\mathbf{n}^2$, the minimum possible value.*

Conjecture 1.1 is considered one of the two most important open problems in algebraic complexity theory, the other being L. Valiant's permanent vs. determinant problem (see e.g., [Lan17]). If true, Conjecture 1.1 would have the surprising consequence that, asymptotically, multiplying two $\mathbf{n} \times \mathbf{n}$ matrices is *almost as easy as* adding them.

In [Str94], Strassen proposed the even more astounding *asymptotic rank conjecture*, stating that the analog of Conjecture 1.1 holds for a larger class of tensors. We introduce some definitions before stating the conjecture precisely.

A tensor $T \in A \otimes B \otimes C$ is *concise* if the three linear maps $T_A : A^* \to B \otimes C$, $T_B : B^* \to A \otimes C$ and $T_C : C^* \to A \otimes B$ are injective. Kronecker products of concise tensors are concise and in particular if T is concise then $T^{\boxtimes N}$ is concise as well. In order to avoid trivialities, we always work with concise tensors.

Definition 1.2. The *rank* of $T \in A \otimes B \otimes C$, denoted $\mathbf{R}(T)$, is the smallest integer r such that $T = \sum_{j=1}^{r} u_j \otimes v_j \otimes w_j$ with $u_j \in A$, $v_j \in B$, $w_j \in C$. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be expressed as a limit (in the Euclidean topology) of tensors of rank r. The *asymptotic rank* of T is $\underset{\sim}{\mathbf{R}}(T) = \lim_{N\to\infty} \mathbf{R}(T^{\boxtimes N})^{1/N} = \lim_{N\to\infty} \underline{\mathbf{R}}(T^{\boxtimes N})^{1/N}$. In [Str87] the limits are shown to exist and to be equal.

If $T \in A \otimes B \otimes C$ is concise, then $\mathbf{R}(T) \geq \underline{\mathbf{R}}(T) \geq \underset{\sim}{\mathbf{R}}(T) \geq \max\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Moreover for $T, T' \in A \otimes B \otimes C$, $\mathbf{R}(T \boxtimes T') \leq \mathbf{R}(T)\mathbf{R}(T')$ and similarly for border rank and asymptotic rank. The latter inequality can be strict for rank and border rank, and it is not known whether it can be strict for asymptotic rank.

Border rank and asymptotic rank are semicontinuous under degeneration. Let $G := GL(A) \times GL(B) \times GL(C)$ and let $T, T' \in A \otimes B \otimes C$ be two tensors such that $T' \in \overline{G \cdot T}$, the orbit closure (equivalently in the Zariski or the Euclidean topology) of the tensor T under the natural action of G. In this case, we say that T' is a degeneration of T; we have $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T)$ and $\underset{\sim}{\mathbf{R}}(T') \leq \underset{\sim}{\mathbf{R}}(T)$.

Therefore Conjecture 1.1 can be stated equivalently for every tensor of $\overline{MaMu_m} := \overline{G \cdot M_{\langle \mathbf{n} \rangle}} \subseteq A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = m = \mathbf{n}^2$. More precisely, Conjecture 1.1 says that if $T \in \overline{MaMu_m}$ is a concise tensor, then $\underset{\sim}{\mathbf{R}}(T) = m$.

For a tensor $T = \sum_{ijk} T^{ijk} a_i \otimes b_j \otimes c_k$ with $\{a_i\}$ basis of A and similarly for $\{b_j\}$ and $\{c_k\}$, the support of T is the set $\mathrm{supp}(T) = \{(i,j,k) : T^{ijk} \neq 0\} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$.

Definition 1.3. Given $m \in \mathbb{N}$, let $[m] := \{0, \dots, m-1\}$. Let $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$. The subset $\mathcal{S}$ is called

- *tight* if there exist injective functions $\tau_A : [\mathbf{a}] \to \mathbb{Z}$, $\tau_B : [\mathbf{b}] \to \mathbb{Z}$ and $\tau_C : [\mathbf{c}] \to \mathbb{Z}$ such that $\tau_A(i) + \tau_B(j) + \tau_C(k)$ is constant on every $(i, j, k) \in \mathcal{S}$; up to shifting the functions, we may always assume that this constant is 0;
- *oblique* if $\mathcal{S}$ is an antichain under the partial ordering on $[\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ induced by the natural ordering on $\mathbb{N}$;
- *free* if any two $(i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{S}$ differ in at least two entries.

A tensor $T \in A \otimes B \otimes C$ is *tight* (resp. *oblique*, resp. *free*) if there exists a choice of bases $\{a_i\}_{i\in[\mathbf{a}]}, \{b_j\}_{j\in[\mathbf{b}]}, \{c_i\}_{k\in[\mathbf{c}]}$ such that the support $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ of T is the given bases is a tight (resp. oblique, resp. free) subset; in this case, the chosen basis is called a tight (resp. oblique, resp. free) basis.

There is a geometric interpretation of tightness: tight tensors are those tensors annihilated by a regular semisimple element of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$, under its natural action on $A \otimes B \otimes C$. We observe that the matrix multiplication tensor is tight (see e.g., the proof of [Str87, Thm. 6.6]), and that every tight tensor is oblique and that every oblique tensor is free. Details on the relationships between these notions are given in Section 2. We are not aware of a similar geometric interpretation for obliqueness and freeness. We pose the following problem:

Problem 1.4. Find geometric, i.e., coordinate free, definitions for obliqueness and freeness.

Strassen's asymptotic rank conjecture states that the asymptotic rank of tight concise tensors is minimal. More precisely:

Conjecture 1.5 (Strassen's Asymptotic Rank Conjecture, [Str94])**.** *Let $T \in A \otimes B \otimes C$ be tight and concise. Then $\underset{\sim}{\mathbf{R}}(T) = \max\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.*

In [BCS97], the authors asked if tightness is needed in Conjecture 1.5:

Problem 1.6 ([BCS97], Problem 15.5)**.** Is $\underset{\sim}{\mathbf{R}}(T) = m$ for all concise $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$?

Fix $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. In this paper, we compare Conjecture 1.1, Conjecture 1.5, their generalizations to the set of oblique and free tensors and finally the one proposed in Problem 1.6. We do this by studying geometric properties of the sets to which the conjectures apply, starting from their dimensions. Let $\overline{Tight}_m$, $\overline{Oblique}_m$ and $\overline{Free}_m$ be the closures (equivalently in the Zariski or Euclidean topology) of the sets of tight, oblique and free tensors respectively. We have

Theorem 1.7. *Let $m \geq 2$ and let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. Then*

(i) if $m = \mathbf{n}^2$, then $\dim \overline{MaMu}_m = 3m^2 - 3m$;

(ii) $\dim \overline{Tight}_m = 3m^2 + \lceil \frac{3}{4}m^2 \rceil - 3m$;

(iii) $\dim \overline{Oblique}_m = 3m^2 + \lceil \frac{3}{4}m^2 \rceil - 3m$;

(iv) $\dim \overline{Free}_m = 4m^2 - 3m$.

The statement of (i) is an immediate consequence of the results of [dG78, Bur15, Ges16]. In order to prove the other statements, in Section 2 we determine the maximal cardinality of a tight, oblique or free support respectively. In particular, we have

Theorem 1.8. *Let $\mathcal{S} \subseteq [m] \times [m] \times [m]$. Then*

(i) if $\mathcal{S}$ is tight then $|\mathcal{S}| \leq \lceil \frac{3}{4}m^2 \rceil$ and the inequality is sharp;

(ii) if $\mathcal{S}$ is oblique then $|\mathcal{S}| \leq \lceil \frac{3}{4}m^2 \rceil$ and the inequality is sharp;

(iii) if $\mathcal{S}$ is free then $|\mathcal{S}| \leq m^2$ and the inequality is sharp.

Notice that by dimension reasons, the inclusions $\overline{MaMu}_m \subseteq \overline{Tight}_m \subseteq \overline{Oblique}_m \subseteq \overline{Free}_m$ are strict except possibly for $\overline{Tight}_m \subseteq \overline{Oblique}_m$. Proposition 2.13 proves that $\overline{Tight}_3 = \overline{Oblique}_3$ and Proposition 2.14 proves that $\overline{Tight}_m \subseteq \overline{Oblique}_m$ is strict for $m \geq 4$.

Moreover, when $m = \mathbf{n}^2$, the standard presentation of the matrix multiplication tensor gives $|\mathrm{supp}(M_{\langle \mathbf{n} \rangle})| = \lfloor \frac{3}{2}m^2 \rfloor$. We mention here the relation with the notion of support rank, introduced in [CU03]: the support rank of the matrix multiplication tensor $M_{\langle \mathbf{n} \rangle}$ is the minimal rank of a tensor having support equal to $\mathrm{supp}(M_{\langle \mathbf{n} \rangle})$.

In Section 3, we provide results on the compressibility of tight tensors. A tensor $T \in A \otimes B \otimes C$ is $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-*compressible* if there exist linear spaces $A' \subset A^*$, $B' \subset B^*$, $C' \subset C^*$, respectively of dimensions $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, such that $T|_{A' \otimes B' \otimes C'} = 0$. A tensor is ρ-*multicompressible* if it is $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible for all $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathbb{N}$ such that $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = \rho$. If T is ρ-multicompressible, then it is ρ'-multicompressible, for all $\rho' \leq \rho$. Compressibility was introduced as a genericity condition in [LM18], where it was shown that a generic tensor in $A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ is not $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible if $m \leq \frac{(\mathbf{a}')^2 + (\mathbf{b}')^2 + (\mathbf{c}')^2 + \mathbf{a}'\mathbf{b}'\mathbf{c}'}{\mathbf{a}' + \mathbf{b}' + \mathbf{c}'}$. In particular a generic tensor is not $(\sqrt{\frac{m}{3}}, \sqrt{\frac{m}{3}}, \sqrt{\frac{m}{3}})$-compressible, see [LM18, Ex. 4.3].

Theorem 1.9. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be a tight tensor. Then T is $(\lceil m/2 \rceil, \lceil m/2 \rceil, \lfloor m/2 \rfloor)$-compressible and similarly permuting the order of the factors.*

The proof of Theorem 1.9 is in §3.1.

Finally, we give results on propagation of symmetries under direct sum and Kronecker product. Let $\Phi : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ be the natural action of G on $A \otimes B \otimes C$; Φ has a 2-dimensional kernel $Z_{A,B,C} = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda\mu\nu = 1\}$, so that $G := (GL(A) \times GL(B) \times GL(C))/Z_{A,B,C}$ can be regarded as a subgroup of $GL(A \otimes B \otimes C)$. The symmetry group of T, denoted G_T, is the stabilizer in G under the action induced by Φ, that is $G_T := \{g \in G : g \cdot T = T\}$. The differential $d\Phi$ of Φ induces a map at the level of Lie algebras: we write $\mathfrak{g}_T$ for the annihilator of T under the action of $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C))/\mathfrak{z}_{A,B,C}$ where $\mathfrak{z}_{A,B,C} \simeq \mathbb{C}^2$ is the Lie algebra of $Z_{A,B,C}$. We have the following result on propagation of symmetries.

Theorem 1.10. *Let $T \in A_1 \otimes B_1 \otimes C_1$ and $S \in A_2 \otimes B_2 \otimes C_2$ be concise tensors. Then*

(i) as subalgebras of $(\mathfrak{gl}(A_1 \oplus A_2) \oplus \mathfrak{gl}(B_1 \oplus B_2) \oplus \mathfrak{gl}(C_1 \oplus C_2))/\mathfrak{z}_{A_1 \oplus A_2, B_1 \oplus B_2, C_1 \oplus C_2}$

$$\mathfrak{g}_{T \oplus S} = \mathfrak{g}_T \oplus \mathfrak{g}_S;$$

(ii) as a subalgebras of $(\mathfrak{gl}(A_1 \otimes A_2) \oplus \mathfrak{gl}(B_1 \otimes B_2) \oplus \mathfrak{gl}(C_1 \otimes C_2))/\mathfrak{z}_{A_1 \otimes A_2, B_1 \otimes B_2, C_1 \otimes C_2}$;

$$\mathfrak{g}_{T \boxtimes S} \supseteq \mathfrak{g}_T \otimes \mathrm{Id}_{A_2 \otimes B_2 \otimes C_2} + \mathrm{Id}_{A_1 \otimes B_1 \otimes C_1} \otimes \mathfrak{g}_S;$$

(iii) if $\mathfrak{g}_T = 0$ *and* $\mathfrak{g}_S = 0$ *then* $\mathfrak{g}_{T\boxtimes S} = 0$.

The containment of (ii) in Theorem 1.10 can be strict, for instance in the case of the matrix multiplication tensor. Additional examples are provided in [CGLV18a]. We propose the following problem.

Problem 1.11. Characterize tensors $T \in A \otimes B \otimes C$ such that $\mathfrak{g}_T \otimes \mathrm{Id}_{A\otimes B\otimes C} + \mathrm{Id}_{A\otimes B\otimes C} \otimes \mathfrak{g}_T$ is strictly contained in $\mathfrak{g}_{T^{\boxtimes 2}} \in A^{\otimes 2} \otimes B^{\otimes 2} \otimes C^{\otimes 2}$.

Context. This paper is part of a larger project to bring methods from algebraic geometry and representation theory to the study of upper complexity bounds for matrix multiplication and tensors in general. Define the *exponent of matrix multiplication* $\omega = \inf\{\tau : \mathbf{R}(M_{\langle \mathbf{n}\rangle}) = O(\mathbf{n}^\tau)\}$, so Conjecture 1.1 states that $\omega = 2$. Starting from Strassen's work of 1969 [Str69], proving $\omega < 2.81$, there has been considerable effort to find upper and lower bounds on ω. The most successful method for finding upper bounds on ω has been Strassen's *laser method* [Str87], which builds on work of Schönhage [Sch81]. In 1989, this method was applied to the Kronecker square of the Coppersmith-Winograd tensor

$$T_{CW,q} = \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) + \\ + a_{q+1} \otimes b_0 \otimes c_0 + a_0 \otimes b_{q+1} \otimes c_0 + a_0 \otimes b_0 \otimes c_{q+1} \in (\mathbb{C}^{q+2})^{\otimes 3}$$

to show the upper bound $\omega < 2.38$ [CW90]. No improvement occurred until 2010-2014, when the laser method was applied to higher Kronecker powers of $T_{CW,q}$, improving the upper bounds ω to the current state of the art $\omega < 2.373$ [Sto10, Wil12, Le 14]. In [AFL15] the authors proved that Strassen's laser method applied to Kronecker powers of the Coppersmith-Winograd tensor (with some additional technical assumptions) cannot prove upper bounds beyond $\omega < 2.3$. Motivated by this result, in [CGLV18b], we uncover new examples of tensors that share the property with $T_{CW,q}$ of having a large dimensional symmetry group. These new tensors, while of interest, are not directly useful for the laser method. However, we observed that they all share the property of having minimal *slice rank* [BCC$^+$17, ST16] (under a natural genericity assumption), a coarsening of the notion of compressibility defined as follows: given $T \in A \otimes B \otimes C$, the *slice rank* of T is

$$SR(T) := \min\{\mathbf{a} + \mathbf{b} + \mathbf{c} - (\mathbf{a}' + \mathbf{b}' + \mathbf{c}') : T \text{ is } (\mathbf{a}', \mathbf{b}', \mathbf{c}') - \text{compressible}\}.$$

This has given rise to a whole new class of tensors which are potentially useful for the laser method. We make a study of such tensors and their potential to prove better upper bounds for ω in the forthcoming [CGLV18a]. The present paper is not devoted to specific tensors useful for the laser method, but rather classes of tensors that have arisen in Strassen's work.

The classes of tight and oblique tensors were introduced by Strassen in [Str91] in his study of the asymptotic spectrum of tensors, and the class of free tensors, first introduced in [Nes84, Fra02], was studied in [CVZ18] in the same context. In [Str91], Strassen addressed the problem of asymptotic degeneration of tensors. Given $T, T' \in A \otimes B \otimes C$, we say that T asymptotically degenerates to T' if there exists $a_N \in o(N)$ such that

$$T'^{\boxtimes N} \in \overline{G \cdot T^{\boxtimes (N+a_N)}}.$$

Asymptotic degeneration defines a preorder on the set $\mathcal{X}$ of *all* 3-tensors (in all finite dimensions). Strassen proved (in a more general context) that this preorder is completely characterized by *spectral points*, namely monotone semi-ring homomorphism from $\mathcal{X}$ to $\mathbb{R}_+$, where the semi-ring

structure on $\mathcal{X}$ is given by direct sum and Kronecker product: in particular, given $T, T' \in \mathcal{X}$, T asymptotically degenerates to T' if and only if $\phi(T) \geq \phi(T')$ for all monotone semi-ring homomorphisms on $\mathcal{X}$. In this setting, the asymptotic rank $\underset{\sim}{\mathbf{R}}(T)$ is the supremum of the value of all spectral points at T. In [Str91], Strassen introduced a class of spectral points, *Strassen's support functionals*, which can be described in terms of compressibility as follows.

Let $\mathcal{F}$ denote the set of triples of increasing complete flags in A, B, C. Let A_i be the i-th vector space in the first flag and similarly for B and C. For $f \in \mathcal{F}$ and $T \in A \otimes B \otimes C$ define

$$\mathrm{Incompr}_f(T) := \{(i,j,k) : T|_{A_i^\perp \otimes B_j^\perp \otimes C_k^\perp} \neq 0\} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}],$$

the incompressible subspaces in the triples of flags. Note that $(i,j,k) \in \mathrm{Incompr}_f(T)$ implies $(i',j',k') \in \mathrm{Incompr}_f(T)$ for all $i' \leq i$, $j' \leq j$, $k' \leq k$.

Given $\theta = (\theta_1, \theta_2, \theta_3)$ such that $\theta_i \geq 0$, $\sum \theta_i = 1$, the Strassen (upper) logarithmic support functional associated to θ is defined as

$$\widehat{\rho}^\theta(T) := \min_{f \in \mathcal{F}} \max \left\{ \sum_{d=1}^3 \theta_d H(p_d) : \begin{array}{l} p \text{ probability distribution on } [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}] \text{ such that} \\ \mathrm{supp}(p) \subseteq \mathrm{Incompr}_f(T) \end{array} \right\}.$$

Here, for $d = 1, 2, 3$, p_d is the d-th marginal distribution of p, and $H(p) = -\sum p_i \log(p_i)$ denotes the Shannon entropy of the distribution p. Strassen showed that $F^\theta(T) = 2^{\widehat{\rho}^\theta(T)}$ is a spectral point on the semi-ring of oblique tensors and conjectured that they are all spectral points on the sub-semi-ring of tight tensors. In particular, since $F^\theta(T) \leq \max\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, this would imply Conjecture 1.5.

In [CVZ18], the first nontrivial examples of spectral points on $\mathcal{X}$ were introduced, and it was shown that they reduce to the ones determined by Strassen on the subclass of free tensors.

In [CGJ18], a refinement of the inequality $\underline{\mathbf{R}}(T) \geq \underset{\sim}{\mathbf{R}}(T)$ was proposed, introducing the "tensor asymptotic rank" $\mathbf{R}^{\otimes}(T)$, a version of asymptotic rank defined with respect to tensor power rather than the Kronecker power: one has $\underline{\mathbf{R}}(T) \geq \mathbf{R}^{\otimes}(T) \geq \underset{\sim}{\mathbf{R}}(T)$.

2. Tight, oblique, and free tensors

In this section, we establish information about the sets of tight, oblique and free tensors, and prove Theorems 1.7 and 1.8.

2.1. **Tight tensors.** Tight tensors can be characterized as follows. Let $T \in A \otimes B \otimes C$. Then T is tight (in the sense of Definition 1.3) if and only if $\mathfrak{g}_T$ contains a regular semisimple element of $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C))/\mathfrak{z}_{A,B,C}$, that is an element that, under some choice of bases, has the form $L = (X, Y, Z)$ with X, Y, Z diagonal matrices, each of them having distinct (rational) eigenvalues. This is shown in [Str91, Str05]. Equivalently, T is stabilized by a regular semisimple one-parameter subgroup of $(GL(A) \times GL(B) \times GL(C))/(\mathbb{C}^*)^2$. Observe that the tightness of T in a given basis only depends on the support of T.

Example 2.1 (A tight support of cardinality $\lceil \frac{3}{4}m^2 \rceil$)**.** Let $m \geq 0$ be an odd integer and write $m = 2\ell + 1$. Define

$$\mathcal{S}_{t\text{-}max,m} = \{(i,j,k) \in [m] \times [m] \times [m] : i + j + k = 3\ell\}.$$

By Definition 1.3, $\mathcal{S}_{t\text{-}max,m}$ is tight. Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be any tensor with support $\mathcal{S}_{t\text{-}max,m}$. Let $L = (U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ be the triple of diagonal

matrices $U = V = W$ having $i - \ell$ at the i-th diagonal entry, with $i = 0, \dots, m-1$. Then $L.T = 0$, because for every element $(i, j, k) \in \operatorname{supp}(T)$ we have

$$L.(a_i \otimes b_j \otimes c_k) = [(i-\ell) + (j-\ell) + (k-\ell)]a_i \otimes b_j \otimes c_k = 0.$$

If T has support $\mathcal{S}_{t\text{-}max,m}$, one can write $T = \sum_{jk} T^{jk} a_{3\ell-j-k} \otimes b_j \otimes c_k$. We can represent T as an $m \times m$ matrix whose entries are elements of A; in this case, we have

$$\begin{bmatrix} & & & & T^{0,\ell}a_{2\ell} & \cdots & T^{0,2\ell-1}a_{\ell+1} & T^{0,2\ell}a_\ell \\ & & & \iddots & & & \iddots & T^{1,2\ell}a_{\ell-1} \\ & & \iddots & & & \iddots & & \vdots \\ & T^{\ell,0}a_{2\ell} & & & \iddots & & & T^{\ell,2\ell}a_0 \\ & \vdots & & \iddots & & & \iddots & \\ T^{2\ell-1,0}a_{\ell+1} & & \iddots & & & \iddots & & \\ T^{2\ell,0}a_\ell & T^{2\ell,1}a_{\ell-1} & \cdots & T^{2\ell,\ell}a_0 & & & & \end{bmatrix} . \tag{1}$$

Each nonzero entry in this matrix corresponds to an element of $\mathcal{S}_{t\text{-}max,m}$; each of the two triangles of 0's (the top left and the bottom right) consists of $\binom{\ell+1}{2}$ entries. Therefore the number of nonzero entries is $(2\ell+1)^2 - (\ell+1)(\ell) = 3\ell^2 + 3\ell + 1 = \lceil \frac{3}{4}m^2 \rceil$.

If $m = 2\ell$ is even, one obtains a tight support of cardinality $3\ell^2 = \lceil \frac{3}{4}m^2 \rceil$ by erasing the last row and the last column of the matrix and setting a_0 to 0. Geometrically this is equivalent to applying the projection which sends $a_0, b_{2\ell}, c_{2\ell}$ to 0 and the other basis vectors of the odd dimensional spaces to basis vectors of the even dimensional spaces. Explicitly, if one has bases $\{a_0, \dots, a_{2\ell-1}\}$, $\{b_0, \dots, b_{2\ell-1}\}$,$\{c_0, \dots, c_{2\ell-1}\}$ of the spaces A, B, C of dimension 2ℓ, the tight support is determined by the functions $\tau_A(i) = i - \ell + 1$, $\tau_B(j) = \tau_C(j) = j - \ell$.

It turns out that the element L introduced in Example 2.1 is, up to scale, the only non-trivial element of $\mathfrak{g}$ which stabilizes a generic tensor with support $\mathcal{S}_{t\text{-}max,m}$, as shown in the following result.

Proposition 2.2. *Let $T \in A \otimes B \otimes C$ be a generic tensor with support $\mathcal{S}_{t\text{-}max,m}$. Then $\dim \mathfrak{g}_T = 1$ and $\mathfrak{g}_T = \langle L \rangle$ where $\langle - \rangle$ denotes the linear span and $L = (U, V, W)$ where U, V, W are diagonal with $u_i^i = v_i^i = w_i^i = i - \ell$.*

Proof. The Theorem of semicontinuity of dimension of the fiber (see e.g., [Sha94, Thm. 1.25]) implies that $\dim \mathfrak{g}_T$ is an upper semicontinuous function. In particular, it suffices to prove the statement for a single element T with support $\mathcal{S}_{t\text{-}max,m}$. Suppose that the coefficients of T are $T^{ijk} = 1$ for every $(i, j, k) \in \mathcal{S}_{t\text{-}max,m}$.

We give the proof in the case $m = 2\ell+1$ odd. If m is even, the argument is essentially the same, with minor modifications to the index ranges.

Let $d\Phi : \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C) \to \mathfrak{gl}(A \otimes B \otimes C)$ be the differential of the map Φ defined in §1. We show that the stabilizer of T under the action of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ has dimension 3, and coincides with $\langle L \rangle + \ker(d\Phi)$.

Let $(U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$; set $u^i_{i'} = 0$ if $i, i' \notin \{0, \dots, 2\ell\}$ and similarly for $v^j_{j'}$ and $w^k_{k'}$. Suppose $(U, V, W) \in \mathfrak{g}_T$, so that every triple (i, j, k) provides a (possibly trivial) equation on the entries of U, V, W as follows

$$(i,j,k) \qquad u^{i'}_i + v^{j'}_j + w^{k'}_k = 0 \tag{2}$$

where i', j', k' are the only integers such that $i' + j + k = i + j' + k = i + j + k' = 3\ell$. Let $\rho = 3\ell - (i + j + k)$; moreover $\rho \in \{-2\ell, \dots, 2\ell\}$ and $\rho = i' - i = j' - j = k' - k$. The equations in (2) can be partitioned into $4\ell + 1$ subsets, indexed by $\rho = -2\ell, \dots, 2\ell$, so that equations in distinct subsets involve disjoint sets of variables. Our goal is to show that the ρ-th set of equations has no nontrivial solutions if $\rho \neq 0$, whereas the 0-th set of equations has exactly a space of solutions of dimension 3 which induces $(U, V, W) \in \langle L \rangle + \ker(d\Phi)$. Indeed, notice that $(U, V, W) \in \langle L \rangle + \ker(d\Phi)$ satisfies all equations in (2).

We consider three separate cases: $\rho = 0, 0 < \rho < \ell$ and $\rho \geq \ell$. The cases $0 > \rho > -\ell$ and $\rho \leq -\ell$ are analogous.

Case $\rho \geq \ell$. First, observe that $u^\rho_0 = v^\rho_0 = w^\rho_0 = 0$. To show this, consider the three equations corresponding to $(i, j, k) = (0, 0, 3\ell - \rho), (0, 3\ell - \rho, 0)$ and $(3\ell - \rho, 0, 0)$, which give the linear system

$$\begin{cases} u^\rho_0 + v^\rho_0 = 0 \\ u^\rho_0 + w^\rho_0 = 0 \\ v^\rho_0 + w^\rho_0 = 0 \end{cases} \tag{3}$$

in the three variables $u^\rho_0, v^\rho_0, w^\rho_0$; this linear system has full rank. This shows $u^i_0 = v^j_0 = w^k_0 = 0$ if $i, j, k \geq \ell$.

Now fix q with $\ell > q \geq 1$; we show that $u^{\rho+q}_q = v^{\rho+q}_q = w^{\rho+q}_q = 0$. The equation corresponding to $(i, j, k) = (q, 0, 3\ell - \rho - q)$ is $u^{\rho+q}_q + v^\rho_0 = 0$, which provides $u^{\rho+q}_q = 0$ since $v^\rho_0 = 0$; similarly $v^{\rho+q}_q = w^{\rho+q}_q = 0$. If $q = \ell$, then $\rho = \ell$ as well (otherwise $u^{\ell+q}_q$ is trivially 0 because $\ell + q > 2\ell$). In this case, the equations corresponding to $(i, j, k) = (0, \ell, \ell), (\ell, 0, \ell), (\ell, \ell, 0)$ provide a linear system similar to (3) which provides $u^{2\ell}_\ell = v^{2\ell}_\ell = w^{2\ell}_\ell = 0$.

Apply a similar argument to the case $\rho \leq -\ell$.

Case $0 < \rho < \ell$. We have $\ell \geq 2$, otherwise this case does not occur. First, we show that $u^{2\ell}_{2\ell-\rho} = v^{2\ell}_{2\ell-\rho} = w^{2\ell}_{2\ell-\rho} = 0$. This is obtained in two steps. First consider the three equations corresponding to the indices $(2\ell - \rho + 1, \ell - 1, 0), (2\ell - \rho + 2, \ell - 2, 0), (2\ell - \rho + 1, \ell - 2, 1)$, which are

$$\begin{aligned} v^{\ell-1+\rho}_{\ell-1} + w^\rho_0 &= 0, \\ v^{\ell-2+\rho}_{\ell-2} + w^\rho_0 &= 0, \\ v^{\ell-2+\rho}_{\ell-2} + w^{\rho+1}_1 &= 0; \end{aligned} \tag{4}$$

these provide $v_{\ell-1}^{\ell-1+\rho}+w_1^{\rho+1}=0$. Now the equation corresponding to the indices $(2\ell-\rho,\ell-1,1)$, namely $u_{2\ell-\rho}^{2\ell}+v_{\ell-1}^{\ell-1+\rho}+w_1^{\rho+1}=0$, reduces to $u_{2\ell-\rho}^{2\ell}=0$; similarly, we have $v_{2\ell-\rho}^{2\ell}=w_{2\ell-\rho}^{2\ell}=0$.

This provides the base case for an induction argument. If $q\geq 1$, we show that $u_{2\ell-q-\rho}^{2\ell-q}=0$. This argument is similar to the one before: the three equations corresponding to $(2\ell-\rho-(q-1),\ell+(q-1),0),(2\ell-\rho+1-(q-1),\ell-1+(q-1),0),(2\ell-\rho-(q-1),\ell-1+(q-1),1)$, together with the induction hypothesis, reduce to $v_{\ell+(q-1)}^{\ell+(q-1)+\rho}+w_1^{\rho+1}=0$. The latter equality, together with the equation corresponding to the indices $(2\ell-q-\rho,\ell+(q-1),1)$ gives $u_{2\ell-q-\rho}^{2\ell-q}=0$. Similarly, we have $v_{2\ell-q-\rho}^{2\ell-q}$ for every $q=0,\dots,2\ell-\rho$. We conclude that $u_i^{i+\rho}=v_j^{j+\rho}=w_k^{k+\rho}=0$ for every i,j,k and every $0<\rho<\ell$.

Apply a similar argument to the case $0>\rho>-\ell$.

Case $\rho=0$. We may work modulo $\ker(d\Phi)=\langle(\mathrm{Id}_A,-\mathrm{Id}_B,0),(\mathrm{Id}_A,0,-\mathrm{Id}_C)\rangle$. In particular, we may assume V,W satisfy $\mathrm{trace}(V)=\mathrm{trace}(W)=0$. Consider all equations (ℓ,j,k) so that $j+k=2\ell$. Adding them up and using the traceless condition, we have $u_\ell^\ell=0$ and therefore $v_{\ell+q}^{\ell+q}=-w_{\ell-q}^{\ell-q}$ for $q=-\ell,\dots,\ell$. Let $\xi=u_{\ell+1}^{\ell+1}$. Then for every q, the equation $(\ell+1,\ell+q-1,\ell-q)$ gives $v_{\ell+q-1}^{\ell+q-1}+\xi=-w^{\ell-q}=v^{\ell+q}$, so that one has $v_{\ell+q}^{\ell+q}=v_\ell^\ell+q\xi$ and similarly $w_{\ell+q}^{\ell+q}=w_\ell^\ell+q\xi$. Since V and W are traceless, we obtain $v_\ell^\ell=w_\ell^\ell=0$ and $v_{\ell+q}^{\ell+q}=w_{\ell+q}^{\ell+q}=q\xi$. In particular, by adding up the equations for the form $(\ell+q,\ell-q,\ell)$ for $q=-\ell,\dots,\ell$, we observe that U is traceless as well, and by a similar argument $u_{\ell-q}^{\ell+q}=q\xi$ as well, so that $(U,V,W)=L$. This shows that modulo $\ker(d\Phi)$ we have a $\mathfrak{g}_T=\langle L\rangle$, and concludes the proof. □

2.2. **Oblique tensors.** Recall that a tensor T is oblique if there are bases such that $\mathrm{supp}(T)$ is an antichain in $[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$.

Remark 2.3. Every tight set is oblique. Let $\mathcal{S}\subseteq[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$ be a tight set. After permuting the elements of $[\mathbf{a}],[\mathbf{b}],[\mathbf{c}]$, we may assume that τ_A,τ_B,τ_C are strictly increasing. Suppose $\mathcal{S}$ is not an antichain in $[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$ and let $(i_1,j_1,k_1),(i_2,j_2,k_2)\in\mathcal{S}$ distinct such that $i_1\leq i_2,j_1\leq j_2,k_1\leq k_2$, with at least one strict inequality. Therefore $\tau_A(i_1)+\tau_B(j_1)+\tau_C(k_1)<\tau_A(i_2)+\tau_B(j_2)+\tau_C(k_2)$, in contradiction with the assumption that $\mathcal{S}$ is tight.

In order to give some insights on oblique subsets, we introduce terminology from [Pro82].

Definition 2.4 ([Pro82])**.** Let $(P,\prec)$ be a poset and let $x,y\in P$. The element x *covers* y if $y\prec x$ and there does not exist $z\in P$ such that $y\prec z\prec x$. A *ranked poset* P of *length* r is a poset P with a partition $P=\bigsqcup_{i=0}^r P_i$ into $r+1$ *ranks* P_i, such that elements in P_i cover only elements in P_{i-1}. A ranked poset of length r is *rank symmetric* if $|P_i|=|P_{r-i}|$ for $1\leq i<r/2$. It is *rank unimodal* if $|P_1|\leq|P_2|\leq\cdots\leq|P_{h_0}|$ and $|P_{h_0}|\geq|P_{h_0+1}|\geq\cdots\geq|P_{r+1}|$, for some $1\leq h_0\leq r+1$.

A poset is *Peck* if it is rank symmetric, rank unimodal and for every $\ell\geq 1$ no union of ℓ antichains contains more elements than the union of the ℓ largest ranks of P.

Example 2.5. For every $\mathbf{a}$, the poset $[\mathbf{a}]$ is ranked of length $\mathbf{a}-1$ and it is Peck.

Using representation-theoretic methods, Proctor [Pro82, Thm. 2] showed that products of Peck posets are Peck posets, with respect to the natural *product ordering* and with rank function defined by the sum of the rank functions of the factors; in particular $[\mathbf{a}]\times[\mathbf{b}]\times[\mathbf{c}]$ is Peck

according to the induced partial ordering on the product and the rank function is given by $h(i,j,k) = i+j+k$.

Remark 2.6. Oblique supports entirely contained in a single rank are tight. More explicitly, let $P = [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$. Every oblique tensor T whose support $\mathcal{S}_T$ is an antichain in some rank P_h of P is tight. In particular $\mathcal{S}_{t\text{-}max,m}$ coincides with $P_{3\ell}$, with $m = 2\ell+1$ or $m = 2\ell$; using Proctor's terminology, this corresponds to the $\mathfrak{sl}_2$-weight space of weight 0 in the representation $\mathbb{C}^P = \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$ where the factors are regarded as irreducible $\mathfrak{sl}_2$-representations.

The following is a slightly stronger version of Theorem 1.8(ii):

Theorem 2.7. *Let* $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ *and let* $\mathcal{S} \subset [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ *be oblique. Then*

$$|\mathcal{S}| \leq \begin{cases} \mathbf{ab} - \lfloor \frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})^2}{4} \rfloor & \text{if } \mathbf{a}+\mathbf{b} \geq \mathbf{c} \\ \mathbf{ab} & \text{if } \mathbf{a}+\mathbf{b} \leq \mathbf{c}. \end{cases}$$

Moreover, in all cases there exist $\mathcal{S}$ *such that equality holds.*

Proof. We follow the proof of [Str87, Thm. 6.6]. Since $P = [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is Peck, the cardinality of a maximal antichain is upper bounded by the maximal rank subset: since a Peck set is unimodular, the maximal rank is the central one, namely $P_{h_{max}} = \{(i,j,k) : i+j+k = h_{max}\}$ where $h_{max} = \lfloor \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-3}{2} \rfloor$ (and equivalently $\lceil \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-3}{2} \rceil$).

If $\mathbf{a}+\mathbf{b} < \mathbf{c}$ then for every $(i,j) \in [\mathbf{a}] \times [\mathbf{b}]$ there exists $k \in [\mathbf{c}]$ such that $i+j+k = h_{max}$, so $|P_{h_{max}}| = \mathbf{ab}$ and the statement of the theorem holds.

Now suppose $\mathbf{a}+\mathbf{b} \geq \mathbf{c}$. Let $\psi : P \to [\mathbf{a}] \times [\mathbf{b}]$ be the projection onto the first two factors. Note that ψ restricted to each P_h is injective because P_h is an antichain. Then $|P_{h_{max}}| = |\psi(P_{h_{max}})|$. We compute the number of elements of $\psi(P_{h_{max}})$. Consider its complement in $[\mathbf{a}] \times [\mathbf{b}]$, that is the set of pairs $(i,j) \in [\mathbf{a}] \times [\mathbf{b}]$ for which there is no $k \in [\mathbf{c}]$ with $i+j+k = h_{max}$. Since $0 \leq k \leq \mathbf{c}-1$, these are exactly pairs (i,j) satisfying one of the following conditions:

(i) $i+j \leq h_{max} - (\mathbf{c}-1) - 1$, that is $i+j \leq \lfloor \frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2} \rfloor$;

(ii) $h_{max} \leq i+j-1$, that is $\lfloor \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-1}{2} \rfloor \leq i+j$.

Notice that (i) and (ii) are mutually exclusive. Let $\theta = \lfloor \frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2} \rfloor$. For every $i = 0, \ldots, \theta$, and every $j = 0, \ldots, \theta - i$, we have $i+j \leq \theta$; this gives $1+2+\cdots+(\theta+1) = \binom{\theta+2}{2}$ pairs (i,j) satisfying condition (i). Now, let $i' = \mathbf{a}-1-i$ and $j' = \mathbf{b}-1-j$: condition (ii) can be rephrased as $\mathbf{a}+\mathbf{b}-2-\lfloor \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}-1}{2} \rfloor \geq i'+j'$ which in turn becomes $i'+j' \leq \eta$ where $\eta = \lceil \frac{\mathbf{a}+\mathbf{b}-\mathbf{c}-3}{2} \rceil$; this provides $\binom{\eta+2}{2}$ pairs (i',j') which correspond to $\binom{\eta+2}{2}$ pairs (i,j) satifying (ii). We conclude that the complement of $\psi(P_{h_{max}})$ in $[\mathbf{a}] \times [\mathbf{b}]$ consists of $\binom{\theta+2}{2} + \binom{\eta+2}{2}$ elements. To conclude, observe $\binom{\theta+2}{2} + \binom{\eta+2}{2} = \lfloor \frac{(\mathbf{a}+\mathbf{b}-\mathbf{c})^2}{4} \rfloor$. □

Choosing $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ in Theorem 2.7, one obtains the bound of Theorem 1.8(ii). Since every tight tensor is oblique, the same bound holds for tight tensors. Since $\mathcal{S}_{t\text{-}max,m}$ from Example 2.1 is a tight support of cardinality $\lceil \frac{3}{4}m^2 \rceil$ (which in fact corresponds to a maximal antichain as observed in Remark 2.6), we obtain that the bound is sharp both in the oblique and in the tight case.

2.3. **Free tensors.** We recall that a subset $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ is free if any two triples (i,j,k), (i',j',k') in $\mathcal{S}$ differ on at least two entries. Directly from the definition, we have the following

Remark 2.8. Every oblique support is free. Let $\mathcal{S}$ be an oblique support and suppose it is not free. Without loss of generality, $\mathcal{S}$ contains two triples of the form $s_1 = (i,j,k_1)$ and $s_2 = (i,j,k_2)$ for some k_1, k_2. But then, if $k_1 \leq k_2$ then $s_1 \leq s_2$ and if $k_2 \leq k_1$ then $s_2 \leq s_1$, therefore $\mathcal{S}$ is not an antichain, providing a contradiction.

Example 2.9 (A free support of cardinality m^2)**.** We obtain a free support of cardinality m^2 by completing the support $\mathcal{S}_{t\text{-}max,m}$ in a circulant way. More precisely, let $m \geq 0$ be odd with $m = 2\ell + 1$. Define

$$\mathcal{S}_{f\text{-}max,m} = \{(i,j,k) : i + j + k \equiv \ell \mod m\} \subseteq [m] \times [m] \times [m].$$

Notice that in the range where $\ell \leq j + k < 3\ell$, then $i = 0, \ldots, 2\ell$ with $i + j + k = 3\ell$, recovering the structure of $\mathcal{S}_{t\text{-}max,m}$.

It is immediate from the definition that the cardinality of a free support is at most m^2: indeed, any m^2+1 elements would have at least two triples (i,j,k) with the same (i,j). This observation, together with Example 2.9, completes the proof of Theorem 1.8(iii).

2.4. **Proof of Theorem 1.7.** We first describe the general construction that will be used in the proof.

Fix a vector space V and let $1 \leq \kappa \leq \dim V - 1$. Let $G(\kappa, V)$ denote the Grassmannian of κ-planes through the origin in V and let $\pi_G : \mathcal{T} \to G(\kappa, V)$ denote the tautological subspace bundle of $G(\kappa, V)$, i.e., the vector bundle whose fiber over a κ-dimensional plane $E \in G(\kappa, V)$ is E itself. Let $\pi_V : \mathcal{T} \to V$ denote the projection to V, that is $\pi_V : (E, v) \to v$ for every $E \in G(\kappa, V)$ and $v \in E \subseteq V$.

If $Z \subset G(\kappa, V)$ is a subvariety of dimension z, then $\dim \pi_G^{-1}(Z) = z + \kappa$. Consequently, $\dim \pi_V(\pi_G^{-1}(Z)) \leq z + \kappa$. The action of a group on V naturally induces an action on $G(\kappa, V)$ and the vector bundle $\mathcal{T}$ can be restricted to orbits and orbit-closures of such an action.

We will use this construction in the setting where $V = A \otimes B \otimes C$, and Z is the $GL(A) \times GL(B) \times GL(C)$-orbit closure of the linear space consisting of all tensors with a given support; we refer to such linear space as the *span of a support*.

The variety $\overline{Tight}_m$, (resp. $\overline{Oblique}_m$, $\overline{Free}_m$) is a union of subvarieties of the form $\pi_V(\pi_G^{-1}(Z))$ with $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E}$ and E is the span of a tight (resp. oblique, free) support in some given bases, regarded as an element of $G(\dim E, A \otimes B \otimes C)$. In particular, we have the following

Lemma 2.10. *We have*

$$\dim \overline{Tight}_m = \max \left\{ \dim \pi_V(\pi_G^{-1}(Z)) : \begin{array}{l} Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \\ \text{for some } E \in G(\kappa, A \otimes B \otimes C) \text{ span of a tight support} \end{array} \right\},$$

and similarly for $\overline{Oblique}_m$ *and* $\overline{Free}_m$.

Proof. Every tight tensor is in the $GL(A) \times GL(B) \times GL(C)$ orbit of a tight tensor in a fixed basis. Moreover, the number of tight supports in a fixed basis is finite. This implies that the irreducible components of the variety $\overline{Tight}_m$ have the form $\pi_V(\pi_G^{-1}(Z))$, where $Z =$

$\overline{GL(A)\times GL(B)\times GL(C)\cdot E}$ for some linear space E which is the span of a non-extendible tight support.

Since the number of supports is finite, $\dim \overline{Tight}_m$ is just the dimension of the largest orbits.

The same holds for $\overline{Oblique}_m$ and $\overline{Free}_m$. □

The following lemma gives the dimension of the orbit closure of the span of a concise free support E. Since from Remark 2.3 every tight support is oblique (up to reordering the bases) and from Remark 2.8 every oblique support is free, the same result applies to tight and oblique supports.

Lemma 2.11. *Let $E \in G(\kappa, A\otimes B\otimes C)$ be the span of a concise free support and let $Z = \overline{GL(A)\times GL(B)\times GL(C)\cdot E} \subseteq G(\kappa, A\otimes B\otimes C)$. Then* $\dim Z = \mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

Proof. We show that the affine tangent space to Z at E in the Plucker embedding of $G(\kappa, A\otimes B\otimes C)$ in $\mathbb{P}\Lambda^\kappa(A\otimes B\otimes C)$ has dimension exactly $\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-(\mathbf{a}+\mathbf{b}+\mathbf{c})+1$; in the following, let $\widehat{G}(\kappa, A\otimes B\otimes C)\subseteq \Lambda^\kappa(A\otimes B\otimes C)$ be the cone over $G(\kappa, A\otimes B\otimes C)$. The affine tangent space to Z at E is $\widehat{T}_E Z = \{(\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)).E\}$, which is naturally a subspace of $\Lambda^\kappa(A\otimes B\otimes C)$. Here E is identified with the element $\bigwedge_{s=1}^{\kappa}(a_{i_s}\otimes b_{j_s}\otimes c_{k_s})\in\widehat{G}(\kappa, A\otimes B\otimes C)$, where $\{(i_s,j_s,k_s): s=1,\dots,\kappa\}$ is the free support defining E.

Let $(X,Y,Z)\in\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C)$ for three $m\times m$ matrices X,Y,Z. If X,Y,Z are diagonal, then $(X,Y,Z).E = E$ up to scale. Thus $\dim\widehat{T}_E Z\leq \mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-(\mathbf{a}+\mathbf{b}+\mathbf{c})+1$. In order to show equality, it suffices to observe that the vectors of the form $(X,Y,Z).E$, where X,Y,Z are three matrices which are all 0 except in a single off-diagonal entry in one of them, are linearly independent and span a subspace of $\Lambda^\kappa(A\otimes B\otimes C)$ which does not contain E; in particular, such subspace has dimension $\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

For every $L=(X,Y,Z)$ having exactly one off-diagonal nonzero entry, we observe that $L.E\neq 0$ and that every summand in the expansion of $L.E$ as sum of basis vectors of $\Lambda^\kappa(A\otimes B\otimes C)$ differs from E in exactly one factor: $L.E\neq 0$ follows immediately by freeness, while the second condition is realized whenever E is spanned by basis vectors. In particular, the subspace of $\Lambda^\kappa(A\otimes B\otimes C)$ generated by the $L.E$'s does not contain E.

The same argument shows that the $L.E$'s are linearly independent. Indeed, suppose L_1, L_2 both have exactly one nonzero entry and suppose that $L_1.E$ and $L_2.E$ both have a summand Θ in their expansion as sum of basis vectors of $\Lambda^\kappa(A\otimes B\otimes C)$. Regard Θ as an element of $\widehat{G}(\kappa, A\otimes B\otimes C)$ (it is the wedge product of a set of basis vectors), namely a coordinate κ-plane in $A\otimes B\otimes C$. There are exactly two basis elements $v=a_{i_0}\otimes b_{j_0}\otimes c_{k_0}$, $v'=a_{i'_0}\otimes b_{j'_0}\otimes c_{k'_0}$ such that $v\in E\setminus\Theta$ and $v'\in\Theta\setminus E$ and two of the three factors of v coincide with the corresponding factors of v'. There is a unique element of $L\in\mathfrak{gl}(A)+\mathfrak{gl}(B)+\mathfrak{gl}(C)$ having exactly one off-diagonal entry such that $L.v=v'$, which guarantees $L=L_1=L_2$. In particular, all the $L.E$'s are linearly independent and this concludes the proof. □

In particular, from Lemma 2.11, one immediately obtains $\dim\pi_G^{-1}(Z)$ when Z is the orbit-closure of the span of a free support $\mathcal{S}$. If $\mathbf{a}=\mathbf{b}=\mathbf{c}=m$, we have

$$\dim\pi_G^{-1}(Z) = 3m^2-3m+|\mathcal{S}|. \tag{5}$$

In order to prove Theorem 1.7, it suffices to determine a tight (resp. oblique, free) support $\mathcal{S}$ such that $|\mathcal{S}| = \lceil\frac{3}{4}m^2\rceil$ (resp. $\lceil\frac{3}{4}m^2\rceil$, m^2) such that the projection $\pi_V : \mathcal{T}|_Z\to A\otimes B\otimes C$

is generically finite-to-one. For the tight and oblique cases, we consider $\mathcal{S} = \mathcal{S}_{t\text{-}max,m}$ from Example 2.1, and for the free case we consider $\mathcal{S} = \mathcal{S}_{f\text{-}max,m}$ from Example 2.9.

Tight and Oblique case. Let $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \subseteq G(\lceil \frac{3}{4}m^2 \rceil, A \otimes B \otimes C)$ where $E = \langle \mathcal{S}_{t\text{-}max,m} \rangle$ is the linear space of tensors supported at $\mathcal{S}_{t\text{-}max,m}$. We prove that the fiber of π_V as a generic point of E is 0-dimensional. From Proposition 2.2, we have $\dim G_T = 1$ and in particular the connected component of the identity in G_T is a 1-parameter subgroup which is diagonal in the fixed basis; let Γ_E be this subgroup.

The fiber over T is the subset of $\mathcal{T}|_Z$ defined by $Y_T = \{(F, T) : F \in Z, T \in F\}$. If $(F, T) \in Y_T$, with $F \neq E$, then $F = gE$ for some $g = (g_A, g_B, g_C) \in GL(A) \times GL(B) \times GL(C)$. At least one of g_A, g_B, g_C is not diagonal in the chosen basis, otherwise $gE = E$. The linear space F is a tight support in the bases $g_A(a_i), g_B(b_j), g_C(c_k)$; in particular the one-parameter subgroup $\Gamma_F = g^{-1}\Gamma_E g$ stabilizes every tensor in F and in particular T. Notice that $\Gamma_F \neq \Gamma_E$, because Γ_F is not diagonal in the bases a_i, b_j, c_k. We deduce $\Gamma_F \subseteq G_T$. Since Γ_F is connected (as it is the image of a continuous map of a connected topological space) and contains the identity, it is contained in the identity component of G_T, in contradiction with Proposition 2.2. This shows that $\pi_V|_{\pi_G^{-1}(Z)}$ is generically finite-to-one.

Free case. Let $Z = \overline{GL(A) \times GL(B) \times GL(C) \cdot E} \subseteq G(m^2, A \otimes B \otimes C)$ where $E = \langle \mathcal{S}_{f\text{-}max,m} \rangle$ is the linear space of tensors supported at $\mathcal{S}_{f\text{-}max,m}$. Let T be a tensor in E such that $\mathrm{supp}(T) \subseteq \mathcal{S}_{t\text{-}max,m} \subseteq \mathcal{S}_{f\text{-}max,m}$. The tensor T is tight and the same argument that we followed in the previous case shows that the fiber of π_V is finite at T. By semicontinuity of dimension of the fibers (see e.g., [Sha94, Thm. 1.25]), π_V has 0-dimensional fiber at the generic point of E and therefore $\pi_V|_{\pi_G^{-1}(Z)}$ is generically finite-to-one.

Via equation (5), we now conclude the proof of Theorem 1.7:

$$\begin{aligned}
\dim \overline{\mathit{Tight}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{t\text{-}max,m}| = \\
&&&= 3m^2 - 3m + \lceil \tfrac{3}{4}m^2 \rceil, \\
\dim \overline{\mathit{Oblique}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{t\text{-}max,m}| = \\
&&&= 3m^2 - 3m + \lceil \tfrac{3}{4}m^2 \rceil, \\
\dim \overline{\mathit{Free}}_m &= \dim \pi_V(\pi_G^{-1}(Z)) = \dim \pi_G^{-1}(Z) &&= 3m^2 - 3m + |\mathcal{S}_{f\text{-}max,m}| = \\
&&&= 3m^2 - 3m + m^2.
\end{aligned}$$

2.5. **Tight, oblique and free in small dimension and inclusions among classes of tensors.** Theorem 1.7 allows us to give some corollaries regarding the sets of tight, oblique and free tensors in small dimension.

Corollary 2.12. *(1) If* $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$*, then* $A \otimes B \otimes C = \overline{\mathit{Free}}_3$ *and* $\overline{\mathit{Tight}}_3$ *is a variety of codimension* 2*.*

(2) If $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$*, then* $\overline{\mathit{Free}}_4$ *is a variety of codimension* 2 *and* $\overline{\mathit{Tight}}_4$ *has codimension* 16*.*

We saw that every tight tensor is oblique and every oblique tensor is free. Observe that the inclusions $\overline{\mathit{Oblique}}_m \subseteq \overline{\mathit{Free}}_m$ is strict since the two varieties have different dimension. The varieties $\overline{\mathit{Tight}}_m$ and $\overline{\mathit{Oblique}}_m$ have the same dimension.

Next we show that $\overline{\mathit{Tight}}_3 = \mathit{Oblique}_3$, whereas the inclusion $\overline{\mathit{Tight}}_m \subseteq \overline{\mathit{Oblique}}_m$ is strict for $m \geq 4$.

Proposition 2.13. *Let* $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$. *Then every concise oblique tensor is tight and in particular* $\overline{Tight}_3 = \overline{Oblique}_3$.

Proof. This statement is proved via a computer calculation. There are 144 maximal antichains in $[3]\times[3]\times[3]$; only 80 of these are concise, in the sense that generic tensors with the corresponding support are concise. The group $\mathfrak{S}_3 \times \mathbb{Z}_2$ acts on $[3] \times [3] \times [3]$, where $\mathfrak{S}_3$ permutes the factors and $\mathbb{Z}_2$ maps (i, j, k) to $(2 - i, 2 - j, 2 - k)$. The induced action on subsets of $[3] \times [3] \times [3]$ preserves tight supports and antichains. In particular, without loss of generality, it suffices to prove the statement for an antichain in each orbit of $\mathfrak{S}_3 \times \mathbb{Z}_2$. There are 13 such orbits. The following are representatives for the orbits:

$$\begin{aligned}
\mathcal{S}_1 &= \{(0,0,2),(0,1,1),(1,0,1),(2,2,0)\},\\
\mathcal{S}_2 &= \{(0,0,2),(0,2,0),(1,1,1),(2,0,0)\},\\
\mathcal{S}_3 &= \{(0,0,2),(0,2,1),(1,1,0),(2,0,1)\},\\
\mathcal{S}_4 &= \{(0,0,2),(0,2,1),(1,2,0),(2,1,1)\},\\
\mathcal{S}_5 &= \{(0,0,2),(0,1,1),(0,2,0),(1,0,1),(2,1,0)\},\\
\mathcal{S}_6 &= \{(0,0,2),(0,1,1),(1,0,1),(1,2,0),(2,1,0)\},\\
\mathcal{S}_7 &= \{(0,0,2),(0,1,1),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_8 &= \{(0,0,2),(0,2,0),(1,1,1),(2,0,1),(2,1,0)\},\\
\mathcal{S}_9 &= \{(0,0,2),(0,2,1),(1,1,1),(2,0,1),(2,2,0)\},\\
\mathcal{S}_{10} &= \{(0,1,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_{11} &= \{(0,0,2),(0,1,1),(0,2,0),(1,0,1),(1,1,0),(2,0,0)\},\\
\mathcal{S}_{12} &= \{(0,0,2),(0,2,1),(1,1,1),(1,2,0),(2,0,1),(2,1,0)\},\\
\mathcal{S}_{13} &= \{(0,1,2),(0,2,1),(1,0,2),(1,1,1),(1,2,0),(2,0,1),(2,1,0)\}.
\end{aligned}$$

For each of these, we provide the functions τ_A, τ_B, τ_C which guarantee tightness. We record the functions in the following table

	$(\tau_A(0),\tau_A(1),\tau_A(2))$	$(\tau_B(0),\tau_B(1),\tau_B(2))$	$(\tau_C(0),\tau_C(1),\tau_C(2))$
$\mathcal{S}_1$	$(-2,-3,1)$	$(2,1,0)$	$(-1,1,0)$
$\mathcal{S}_2$	$(1,-2,2)$	$(-1,1,0)$	$(-1,1,0)$
$\mathcal{S}_3$	$(-1,2,-2)$	$(1,2,0)$	$(-4,1,0)$
$\mathcal{S}_4$	$(-2,2,-1)$	$(2,-1,0)$	$(-2,2,0)$
$\mathcal{S}_5$	$(-1,-3,2)$	$(1,-1,2)$	$(-1,2,0)$
$\mathcal{S}_6$	$(0,-1,2)$	$(0,-1,2)$	$(-1,1,0)$
$\mathcal{S}_7$	$(2,-3,1)$	$(2,-3,1)$	$(2,1,0)$
$\mathcal{S}_8$	$(2,-2,1)$	$(-2,1,0)$	$(-2,1,0)$
$\mathcal{S}_9$	$(0,1,2)$	$(0,1,2)$	$(-4,-2,0)$
$\mathcal{S}_{10}$	$(-2,2,1)$	$(-2,1,0)$	$(-2,1,0)$
$\mathcal{S}_{11}$	$(-2,1,4)$	$(-2,1,4)$	$(-2,1,4)$
$\mathcal{S}_{12}$	$(-2,1,4)$	$(-2,1,4)$	$(-5,-2,4)$
$\mathcal{S}_{13}$	$(-1,0,1)$	$(-1,0,1)$	$(-1,0,1)$

This shows that every oblique support in $[3]\times[3]\times[3]$ is tight; in particular, every oblique tensor is tight and $\overline{Tight}_3 = \overline{Oblique}_3$. □

Proposition 2.14. *Let $T \in A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$ be the tensor*

$$\begin{aligned} T =& a_0 \otimes b_2 \otimes c_3 + a_0 \otimes b_3 \otimes c_2 + a_1 \otimes b_0 \otimes c_3 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 \\ &+ a_1 \otimes b_3 \otimes c_0 + a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_0 + a_3 \otimes b_0 \otimes c_2 + a_3 \otimes b_1 \otimes c_0. \end{aligned}$$

Then T is oblique and not tight.

Proof. The proof of obliqueness is directly by observing that the support

$$\mathcal{S} = \{(0,2,3),(0,3,2),(1,0,3),(1,1,2),(1,2,1),(1,3,0),(2,1,1),(2,2,0),(3,0,2),(3,1,0)\}$$

is an antichain in $[4] \times [4] \times [4]$.

On the other hand T is not tight: a direct calculation shows that its annihilator $\mathfrak{g}_T$ is trivial. □

Relying on the additivity result of Theorem 1.10(i), which will be proved in Section 4, one obtains that the inclusion $\overline{Tight}_m \subseteq \overline{Oblique}_m$ is strict for every $m \geq 4$. Let T_4 be the tensor of Proposition 2.14 and define $T_m = T_4 \oplus M_{\langle 1 \rangle}^{\oplus m-4}$. Then T_m is oblique but it is not tight.

We conclude this section with a result on border rank of tight tensors. Let $Seg : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C \to \mathbb{P}(A \otimes B \otimes C)$, $Seg([u],[v],[w]) = [u \otimes v \otimes w]$, be the *Segre embedding*, whose image $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the variety of rank one tensors. Let $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \mathbb{P}(A \otimes B \otimes C)$ be the r-th *secant variety* of $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, that is the variety of tensors of border rank at most r.

Proposition 2.15. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$. Then*

- $\sigma_m(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \overline{Tight}_m$;
- $\sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \not\subseteq \overline{Tight}_m$.

Proof. If $r \leq m$, then $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = \overline{(GL(A) \times GL(B) \times GL(C)) \cdot M_{\langle 1 \rangle}^{\oplus r}}$, where $M_{\langle 1 \rangle}^{\oplus r} = \sum_0^{r-1} a_i \otimes b_i \otimes c_i$. Since $M_{\langle 1 \rangle}^{\oplus r}$ is tight, we have $\sigma_m(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subseteq \overline{Tight}_m$.

Let $T_{std,m} = M_{\langle 1 \rangle}^{\oplus m} + (\sum_1^m a_i) \otimes (\sum_1^m b_i) \otimes (\sum_1^m c_i)$. From the expression one sees that $T_{std,m} \in \sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$. A direct calculation shows that the annihilator $\mathfrak{g}_{T_{std,m}}$ is trivial, therefore $T_{std,m}$ is not tight. We conclude $\sigma_{m+1}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \not\subseteq \overline{Tight}_m$. □

3. Compressibility of Tight tensors

In this section, we study the property of compressibility in the setting of tight tensors. The first subsection is devoted to the proof of Theorem 1.9, the second one to the study of multi-compressibility and the third subsection gives some insights on the combinatorial geometry of tightness.

Given a tight subset $\mathcal{S} \subseteq [\mathbf{a}] \times [\mathbf{b}] \times [\mathbf{c}]$ let τ_A, τ_B, τ_C be the functions defining tightness in Definition 1.3(i). Recall that by Remark 2.3, we may assume that τ_A, τ_B, τ_C are increasing and they add up to 0 on $\mathcal{S}$.

3.1. **Proof of Theorem 1.9.** For this proof it is convenient to have basis indices in $\{1,\dots,m\}$. As above, consider T expressed in a tight basis and let $\tau_A,\tau_B,\tau_C;\{1,\dots,m\}\to\mathbb{Z}$ be the corresponding increasing injective functions, with $\tau_A+\tau_B+\tau_C$ identically 0 on $\operatorname{supp}(T)$. We impose one additional normalization on τ_A,τ_B,τ_C as follows: we assume $\tau_A(\lfloor m/2\rfloor)=\tau_B(\lceil m/2\rceil)=-1$ and if $\tau_C(k)\geq 0$ then $\tau_C(j)\geq 2$: in order to do this, redefine $\tau_A=3\tau'_A-3\tau_A(\lfloor m/2\rfloor)-1$, $\tau'_B=\tau_B-\tau_B(\lceil m/2\rceil)-1$ and $\tau'_C=\tau_C+\tau_A(\lfloor m/2\rfloor)+\tau_B(\lceil m/2\rceil)+2$. Notice that $\tau_A+\tau_B+\tau_C=0$ if and only if $\tau'_A+\tau'_B+\tau'_C=0$, so τ_A,τ_B,τ_C define the same tight support as τ'_A,τ'_B,τ'_C; moreover τ'_A,τ'_B,τ'_C are increasing, $\tau_A(\lfloor m/2\rfloor)=\tau_B(\lceil m/2\rceil)=-1$ and if $\tau_C(k)\geq 0$ then $\tau_C(k)\geq 2$ because $\tau_C\equiv 2 \mod 3$. In fact $\tau_A,\tau_B,\tau_C\equiv 2 \mod 3$ and in particular they are never 0.

Now, we consider two cases:

(i) if $\tau_C(\lceil m/2\rceil)>0$, then choose $A'=\langle\alpha_i : i\in\{\lfloor m/2\rfloor+1,\dots,m\}\rangle$, $B'=\langle\beta_j : j\in\{\lceil m/2\rceil,\dots,m\}\rangle$ and $C'=\langle\{\gamma_k : k\in\{\lceil m/2\rceil,\dots,m\}\rangle$. Notice that $\dim A'=\lceil m/2\rceil$, $\dim B'=\lfloor m/2\rfloor+1$ and $\dim C'=\lfloor m/2\rfloor+1$; moreover, the sum of τ'_A,τ'_B,τ'_C on the product of these subsets is lower bounded by $\tau_A(\lfloor m/2\rfloor+1)+\tau_B(\lceil m/2\rceil)+\tau_C(\lfloor m/2\rfloor)\geq 2-1+2=3>0$. This shows $T|_{A'\otimes B'\otimes C'}=0$ because no elements of $\operatorname{supp}(T)$ appear in this range. In this case T is $(\lceil m/2\rceil,\lfloor m/2\rfloor+1,\lfloor m/2\rfloor+1)$-compressible, and in particular $(\lfloor m/2\rfloor,\lceil m/2\rceil,\lceil m/2\rceil)$-compressible.

(ii) if $\tau_C(\lceil m/2\rceil)<0$, then choose $A'=\langle\alpha_i : i\in\{1,\dots,\lfloor m/2\rfloor\}\rangle$, $B'=\langle\beta_j : j\in\{1,\dots,\lceil m/2\rceil\}\rangle$ and $C'=\langle\{\gamma_k : k\in\{1,\dots,\lceil m/2\rceil\}\rangle$. Notice that $\dim A'=\lfloor m/2\rfloor$, $\dim B'=\lceil m/2\rceil$ and $\dim C'=\lceil m/2\rceil$; moreover, the sum of τ'_A,τ'_B,τ'_C on the product of these subsets is upper bounded by $\tau_A(\lfloor m/2\rfloor)+\tau_B(\lceil m/2\rceil)+\tau_C(\lceil m/2\rceil)\leq -1-1-1=-3$. This shows $T|_{A'\otimes B'\otimes C'}=0$ because no elements of $\operatorname{supp}(T)$ appear in this range. In this case T is $(\lfloor m/2\rfloor,\lceil m/2\rceil,\lceil m/2\rceil)$-compressible.

3.2. **Multicompressibility.** One can discuss a restricted form of multicompressibility, by letting only the dimensions of two factors vary. In this context we have the following result:

Proposition 3.1. *Let $T\in A\otimes B\otimes C$ with $\mathbf{a}=\mathbf{b}=\mathbf{c}=m$. For every $\mathbf{b}',\mathbf{c}'$ with $\mathbf{b}'+\mathbf{c}'\leq m-\lceil\sqrt{m-1}\rceil$, T is $(1,\mathbf{b}',\mathbf{c}')$-compressible and similarly permuting the roles of the three factors.*

Proof. The result is immediate if T is not concise. Suppose T is concise. Let $\sigma_r(\mathbb{P}B\times\mathbb{P}C)\subseteq\mathbb{P}(B\otimes C)$ denote the subvariety of rank at most r elements in $\mathbb{P}(B\otimes C)$. We have $\dim\dim(\sigma_r(Seg(\mathbb{P}B\times\mathbb{P}C)))=2rm-r^2-1$. By conciseness, the image of the flattening map $T_A:A^*\to B\otimes C$ has dimension m, so its projectivization $\mathbb{P}(T_A(A^*))$ intersects $\sigma_r(Seg(\mathbb{P}B\times\mathbb{P}C))$ when $r=m-\lceil\sqrt{m-1}\rceil$.

So fix $r=m-\lceil\sqrt{m-1}\rceil$ and let $\alpha\in A^*$ such that $[T_A(\alpha)]\in\mathbb{P}(T_A(A^*))\cap\sigma_r(Seg(\mathbb{P}B\times\mathbb{P}C))$. Then $T_A(\alpha)$ has rank at most $r=m-\lceil\sqrt{m-1}\rceil$.

Choose bases such that $T_A(\alpha)=b_1\otimes c_1+\cdots+b_r\otimes c_r$ and let $A'=\langle\alpha\rangle$, $B'=\langle\beta_1,\dots,\beta_{\mathbf{b}'}\rangle$ and $C'=\langle\gamma_{\mathbf{b}'+1},\dots,\gamma_{\mathbf{b}'+\mathbf{c}'}\rangle$. Then $\dim A'=1$, $\dim B'=\mathbf{b}'$ and $\dim C'=\mathbf{c}'$; we have $T|_{A'\otimes B'\otimes C'}=0$, so T is $(1,\mathbf{b}',\mathbf{c}')$-compressible. □

We show that tight tensors with support equal to $\mathcal{S}_{t\text{-}max,m}$ are highly multicompressible.

Proposition 3.2. *Let $\mathbf{a}=\mathbf{b}=\mathbf{c}=m$ and let $T\in A\otimes B\otimes C$ be a tight tensor with support $\mathcal{S}_{t\text{-}max,m}$. Then T is $(3\lfloor m/2\rfloor+1)$-multicompressible.*

Proof. Recall from Example 2.1, $\tau_A, \tau_B, \tau_C : [m] \to \mathbb{Z}$, with $\tau_A(i) = \tau_B(i) = \tau_C(i) = i - \ell$ if $m = 2\ell + 1$ is odd and with $\tau_A(i) = .i - \ell + 1$, $\tau_B(j) = \tau_C(j) = j - \ell$ if $m = 2\ell$ is even.

Fix $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 3\lfloor m/2 \rfloor + 1 = 3\ell + 1$. We determine $A' \subseteq A^*, B' \subseteq B^*, C' \subseteq C^*$ with $\dim A' = \mathbf{a}'$, $\dim B' = \mathbf{b}'$, $\dim C' = \mathbf{c}'$ such that $T|_{A' \otimes B' \otimes C'} = 0$. Let $A' = \langle \alpha^i : i \in [\mathbf{a}'] \rangle$, $B' = \langle \beta^j : j \in [\mathbf{b}'] \rangle$,$C' = \langle \gamma^k : k \in [\mathbf{c}'] \rangle$.

We claim that $T|_{A' \otimes B' \otimes C'} = 0$. This follows from the fact that $[\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}'] \cap \mathcal{S}_{t\text{-}max,m} = \emptyset$. If $(i, j, k) \in [\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}']$, we have $\tau_A(i) + \tau_B(j) + \tau_C(k) \leq i - \ell + 1 + j - \ell + k - \ell \leq \mathbf{a}' - 1 + \mathbf{b}' - 1 + \mathbf{c}' - 1 - 3\ell \leq 3\ell + 1 - 2 - 3\ell = -1$ (here the first inequality is in fact an equality if m is even). In particular, there are no elements $(i, j, k) \in [\mathbf{a}'] \times [\mathbf{b}'] \times [\mathbf{c}']$ such that $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$. □

However, we observe that highly multicompressible tensors are not necessarily tight:

Example 3.3. This is an example of a $3\lfloor m/2 \rfloor$-multicompressible tensor that is not tight. Let

$$\begin{aligned} T =& a_0 \otimes b_0 \otimes c_0 + a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 + \\ &+ (a_0 + a_1 + a_2 + a_3) \otimes (b_0 + b_1) \otimes (c_2 + c_3) + (a_1 + a_2 + a_3) \otimes b_2 \otimes (c_2 + c_3) + \\ &+ (a_1 + a_2 + a_3) \otimes b_3 \otimes c_3 + (a_2 + a_3) \otimes b_3 \otimes c_2 \end{aligned}$$

be a tensor in $A \otimes B \otimes C$ with $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$. A direct calculation shows that T has trivial annihilator $\mathfrak{g}_T$, therefore it is not tight. It is easy to verify that T is 6-compressible.

Taking direct sums of copies of the tensor above one obtains highly compressible, not tight tensors in higher dimensions.

More generally, we show that maximally compressible tensors (in the sense of [LM17]) are also highly multicompressible.

Proposition 3.4. *Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$ and let $T \in A \otimes B \otimes C$ be $(m-1, m-1, m-1)$-compressible. Then T is $2m-1$-multicompressible.*

Proof. After fixing bases in A, B, C, we may assume without loss of generality that $T|_{a_0^\perp \otimes b_0^\perp \otimes c_0^\perp} = 0$; in particular T can be written as $T = a_0 \otimes M_A + b_0 \otimes M_B + c_0 \otimes M_C$ where $M_A \in B \otimes C$ and similarly M_B, M_C (and reordering the factors in the second and third summand). Let $\mathbf{a}', \mathbf{b}', \mathbf{c}' \leq m$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 2m - 1$. Moreover, we may assume $M_A \in b_0^\perp \otimes c_0^\perp$ because expressing T in the fixed basis, we can include summands including b_0, c_0 in $b_0 \otimes M_B + c_0 \otimes M_C$ and in fact we may fix bases so that $M_A = \sum_1^r b_i \otimes c_i$ for some $r \leq m - 1$.

If $\mathbf{a}', \mathbf{b}', \mathbf{c}' < m$, let $A' \subseteq a_0^\perp, B' \subseteq b_0^\perp, C' \subseteq c_0^\perp$, so $T|_{A' \otimes B' \otimes C'} = 0$.

Suppose $\mathbf{a}' = m$; therefore $A' = A^*$ and $\mathbf{b}' + \mathbf{c}' = m - 1$. Let $B' = \langle \beta^1, \ldots, \beta^{\mathbf{b}'} \rangle$ and $C' = \langle c^{\mathbf{b}'+1}, \ldots, c^{m-1} \rangle$. Then $T|_{A' \otimes B' \otimes C'} = 0$. This concludes the proof. □

Proposition 3.4 implies in particular that the Coppersmith-Winograd tensors $T_{cw,q}$ and $T_{CW,q}$ are respectively $(2q+1)$-multicompressible and $(2q+3)$-multicompressible respectively.

Regarding the matrix multiplication tensor, we have the following

Remark 3.5. Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{n}^2$ and consider $M_{\langle \mathbf{n} \rangle} \in A \otimes B \otimes C$. Then $M_{\langle \mathbf{n} \rangle} \in A \otimes B \otimes C$ is $3\lfloor \mathbf{n}^2/2 \rfloor$-multicompressible. The proof is similar to that of Proposition 3.4. After a change of basis the flattening map $M_{\langle \mathbf{n} \rangle} : A^* \to B \otimes C$ can be written as a $(\mathbf{n} \times \mathbf{n})$-block diagonal matrix

of linear forms on A, whose diagonal blocks are all equal to the matrix (α^i_j), see [Lan17, Exercise 2.1.7.4]. In this form, it is easy to see that $M_{\langle \mathbf{n} \rangle}$ is $(\mathbf{n}^2, \mathbf{b}', \mathbf{c}')$-compressible for every $(\mathbf{b}', \mathbf{c}')$ with $\mathbf{b}' + \mathbf{c}' = \mathbf{n}^2$.

At this point, consider $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ with $\mathbf{a}' + \mathbf{b}' + \mathbf{c}' = 3\lfloor \mathbf{n}^2/2 \rfloor$. Notice that $(\mathbf{a}' + \mathbf{b}') + (\mathbf{b}' + \mathbf{c}') + (\mathbf{a}' + \mathbf{c}') \leq 3\mathbf{n}^2$, so at least one among $(\mathbf{a}' + \mathbf{b}'), (\mathbf{b}' + \mathbf{c}'), (\mathbf{a}' + \mathbf{c}')$ is bounded from above by $\mathbf{n}^2$. Suppose $\mathbf{b}' + \mathbf{c}' \leq \mathbf{n}^2$. From the argument above $M_{\langle \mathbf{n} \rangle}$ is $(\mathbf{n}^2, \mathbf{b}', \mathbf{c}')$-compressible and therefore $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible.

Finally, recall from [LM18] that when $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, every $T \in A \otimes B \otimes C$ with border rank r is $(3m-r)$-multicompressible. For instance, the tensor $T_{std,m}$ defined in the proof of Proposition 2.15 is $(2m-1)$-multicompressible.

We conclude this section providing a pictorial interpretation of tightness.

3.3. **Combinatorial geometry of tight sets and compressibility.** The three functions $\tau_A, \tau_B, \tau_C : [m] \to \mathbb{Z}$ define a line arrangement in $\mathbb{R}^2$ as follows. Consider in $\mathbb{R}^3$ (with coordinates (x, y, z)) the following arrangement of planes, consisting of the union of three families of parallel planes, each of them comprising m planes:

$$\widehat{\mathcal{A}} = \bigsqcup_{i=0}^{m-1} \{x = \tau_A(i)\} \cup \bigsqcup_{j=0}^{m-1} \{y = \tau_B(j)\} \cup \bigsqcup_{k=0}^{m-1} \{z = \tau_C(k)\}. \tag{6}$$

Let $\mathcal{A}$ be the intersection of $\widehat{\mathcal{A}}$ with the plane $\Pi = \{x+y+z = 0\} \subset \mathbb{R}^3$, which is an arrangement of three families of parallel lines, each consisting of m lines: with respect to coordinates x, y in Π, the three families of lines are $\mathcal{A} = \bigsqcup_{i=0}^{m-1} \{x = \tau_A(i)\} \cup \bigsqcup_{j=0}^{m-1} \{y = \tau_B(j)\} \cup \bigsqcup_{k=0}^{m-1} \{x+y = -\tau_C(k)\}$; we say that the *$x$-direction* of $\mathcal{A}$ is the union of the lines with constant x, the y-direction is the union of lines with constant y and the *z-direction* is the union of lines with slope -1. A subset of lines $\mathcal{A}' \subseteq \mathcal{A}$ is called a subarrangement if it contains at least one line in each direction.

The set $\{p \in \Pi : p \text{ belongs to exactly two lines in } \mathcal{A}\}$ is called the *set of double intersection points* of $\mathcal{A}$. The set $\mathfrak{J}(\mathcal{A}) := \{p \in \Pi : p \text{ belongs to three lines in } \mathcal{A}\}$ is the set of *joints* in $\mathcal{A}$.

Lemma 3.6. *Let T be a tight tensor in a tight basis and let $\mathcal{A}$ be the corresponding arrangement of lines. Then $(\tau_A, \tau_B, \tau_C) : [m]^{\times 3} \to \mathbb{R}^3$ maps $\operatorname{supp}(T)$ bijectively to a subset of $\mathfrak{J}(\mathcal{A})$. In particular, if $\mathcal{A}$ is an arrangement with $\mathfrak{J}(A) = \emptyset$ then $T = 0$.*

Proof. If $(i, j, k) \in \operatorname{supp}(T)$ then $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$, so $(\tau_A(i), \tau_B(j), \tau_C(k)) \in \Pi$ is a point of $\mathcal{A}$ lying on three lines. In particular $(\tau_A(i), \tau_B(j), \tau_C(k)) \in \mathfrak{J}(\mathcal{A})$. □

If T is a tight tensor in a tight basis and $\mathcal{A}$ is the corresponding line arrangement with $\operatorname{supp}(T) \subseteq \mathfrak{J}(\mathcal{A})$, we say that T is *supported* on $\mathcal{A}$. Properties of the support of a tight tensor in a tight basis can be translated into geometric and combinatorial properties of $\mathcal{A}$. For instance, compressibility in given coordinates can be studied combinatorially as follows.

Lemma 3.7. *Let T be a tight tensor and let $\mathcal{A}$ be the corresponding line arrangement. If there exists a sub-arrangement $\mathcal{A}'$ of $\mathcal{A}$ consisting of $\mathbf{a}'$ lines in the x direction, $\mathbf{b}'$ lines in the y direction, and $\mathbf{c}'$ lines in the z direction with $\mathfrak{J}(\mathcal{A}') = \emptyset$, then T is $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$-compressible.*

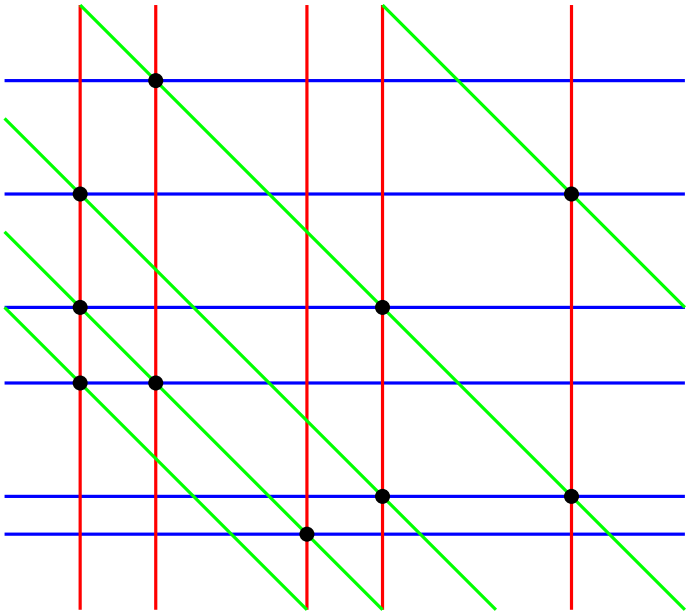

FIGURE 1. An arrangement of lines on the plane Π: the red lines are in the x-direction, the blue lines in the y-direction and the green lines in the z-direction. The joints are marked with black dots.

Proof. After the identification of A, B, C with their duals determined by the choice of bases, let $A' \subseteq A^*$ be the subspace spanned by the basis elements $\{\alpha^i : \{x = \tau_A(i)\} \in \mathcal{A}'\}$ and similarly B' and C'. Then $T' = T|_{A'\otimes B'\otimes C'}$ is tight, and the corresponding arrangement is $\mathcal{A}'$. Since $\mathfrak{J}(\mathcal{A}') = \emptyset$, we conclude by Lemma 3.6. □

4. Propagation of symmetries

This section is devoted to the proof of Theorem 1.10. We use the summation convention for which repeated upper and lower indices are to be summed over. The range of the indices is omitted as they should be clear from the context.

4.1. **Proof of Theorem 1.10(i).** Let $T \in A_1 \otimes B_1 \otimes C_1$ and $S \in A_2 \otimes B_2 \otimes C_2$. Fix bases of $A_1, B_1, C_1, A_2, B_2, C_2$ and write $T = T^{i_1 j_1 k_1} a_{i_1}^{(1)} \otimes b_{j_1}^{(1)} \otimes c_{k_1}^{(1)}$ and $S = S^{i_2 j_2 k_2} a_{i_2}^{(2)} \otimes b_{j_2}^{(2)} \otimes c_{k_2}^{(2)}$. Let $L = (U, V, W) \in \mathfrak{gl}(A_1 \oplus A_2) \oplus \mathfrak{gl}(B_1 \oplus B_2) \oplus \mathfrak{gl}(C_1 \oplus C_2)$. We want to prove that if $L.(T \oplus S) = 0$, then for $\ell = 1, 2$, there is $L_\ell \in \mathfrak{gl}(A_\ell) \oplus \mathfrak{gl}(B_\ell) \oplus \mathfrak{gl}(C_\ell)$ such that $L = L_1 + L_2$ with $L_1.T = 0$ and $L_2.S = 0$. Write $X = X_{11} + X_{12} + X_{21} + X_{22}$ where $X_{11} \in \mathrm{Hom}(A_1, A_1)$ and similarly for the other summands. Consider $X_{21}(T \boxtimes S) = X_{21}(T)$: this is an element of $A_2 \otimes B_1 \otimes C_1$. No other summand of X, nor Y or Z generate a nonzero component in this space. Therefore, $X_{21}(T) = 0$ and by conciseness we deduce $X_{21} = 0$. Similarly $X_{12} = 0$ so that $X = X_{11} + X_{22} \in \mathfrak{gl}(A_1) \oplus \mathfrak{gl}(A_2)$ and similarly for Y and Z. For $\ell = 1, 2$, let $L_\ell = (X_{\ell\ell}, Y_{\ell\ell}, Z_{\ell\ell})$. Then $L = L_1 + L_2$ and $L.(T \boxtimes S) = L_1.T + L_2.S$; notice $L_1.T \in A_1 \otimes B_1 \otimes C_1$ and $L_2.S \in A_2 \otimes B_2 \otimes C_2$ are linearly independent, so if $L.(T \boxtimes S) = 0$, we have $L_1 \in \mathfrak{g}_T$ and $L_2 \in \mathfrak{g}_S$. This concludes the proof.

4.2. **Proof of Theorem 1.10(ii).** This is a straightforward consequence of the Leibniz rule. In general if $\mathfrak{g}_1$ acts on a space V_1 and $\mathfrak{g}_2$ acts on a space V_2, then the action of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on $V_1 \otimes V_2$ is given by the Leibniz rule via $(L_1, L_2) \mapsto L_1 \otimes \mathrm{Id}_{V_2} + \mathrm{Id}_{V_1} \otimes L_2$. If $v_1 \in V_1$ is annihilated by $\mathfrak{g}_1$, and $v_2 \in V_2$ is annihilated by v_2, then $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ annihilates $v_1 \otimes v_2$ via the induced action.

4.3. **Proof of Theorem 1.10(iii).** Fix bases in all spaces. Let $L = (U, V, W) \in \mathfrak{gl}(A_1 \otimes A_2) \otimes \mathfrak{gl}(B_1 \otimes B_2) \otimes \mathfrak{gl}(C_1 \otimes C_2)$ and write U as an $\mathbf{a}_1\mathbf{a}_2 \times \mathbf{a}_1\mathbf{a}_2$ matrix $u^{i_1'i_2'}_{i_1i_2}$, and similarly for V and W. Our goal is to prove that if $L.(T_1 \boxtimes T_2) = 0$, then $L \in \mathfrak{z}_{A_1\otimes A_2, B_1\otimes B_2, C_1\otimes C_2}$.

Write $T_1 = T^{ijk} a^{(1)}_i \otimes b^{(1)}_j \otimes c^{(1)}_k$ and $T_2 = S^{i'j'k'} a^{(2)}_{i'} \otimes b^{(2)}_{j'} \otimes c^{(2)}_{k'}$. The equations for the symmetry Lie algebra $\mathfrak{g}_{T_1\boxtimes T_2}$ is $L.(T_1 \boxtimes T_2) = 0$; in coordinates, for every $i_1, i_2, j_1, j_2, k_1, k_2$, we have (using the summation convention)

$$u^{i_1i_2}_{i_1'i_2'} T^{i_1'j_1k_1} S^{i_2'j_2k_2} + v^{j_1j_2}_{j_1'j_2'} T^{i_1j_1'k_1} S^{i_2j_2'k_2} + w^{k_1k_2}_{k_1'k_2'} T^{i_1j_1k_1'} S^{i_2j_2k_2'} = 0 \tag{7}$$

Let $U(i_1j_1k_1) \in \mathrm{Hom}(A_2, A_2)$ be the matrix whose (i_2, i_2')-th entry is $u^{i_1i_2}_{i_1'i_2'} T^{i_1'j_1k_1}$ and similarly $V(i_1j_1k_1)$ and $W(i_1j_1k_1)$. Let $L(i_1j_1k_1) = (U(i_1j_1k_1), V(i_1j_1k_1), W(i_1j_1k_1))$. From (7), we have $L(i_1j_1k_1).T_2 = 0$, namely $L(i_1j_1k_1) = \mathfrak{z}_{A_2,B_2,C_2}$, so that $U(i_1j_1k_1) = u(i_1j_1k_1)\mathrm{Id}_{A_2}, V(i_1j_1k_1) = v(i_1j_1k_1)\mathrm{Id}_{B_2}, W(i_1j_1k_1) = w(i_1j_1k_1)\mathrm{Id}_{C_2}$, with $u(i_1j_1k_1) + v(i_1j_1k_1) + w(i_1j_1k_1) = 0$.

In particular, if $i_2 \neq i_2'$, we have $U(i_1j_1k_1)^{i_2}_{i_2'} = 0$ for every i_1, j_1, k_1, which by definition provides $u^{i_1i_2}_{i_1'i_2'} T^{i_1'j_1k_1} = 0$. This implies that the $\mathbf{a}_1 \times \mathbf{a}_1$ matrix $(u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'}$ satisfies $((u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'}, 0, 0) \in \mathfrak{g}_{T_1}$. By conciseness, this implies $(u^{i_1i_2}_{i_1'i_2'})_{i_2,i_2'} = 0$, and therefore $u^{i_1i_2}_{i_1'i_2'} = 0$ for every i_1, i_1' and every $i_2 \neq i_2'$. By exchanging the role of the two tensors, we obtain that $u^{i_1i_2}_{i_1'i_2'} = 0$ for every $i_1 \neq i_1'$ and every i_2, i_2'. We deduce that U is diagonal. Similar argument applies to V and W.

Consequently, for each fixed $i_1, i_2, j_1, j_2, k_1, k_2$, (7) reduces to (with no summation) $T^{i_1j_1k_1} S^{i_2j_2k_2} (u^{i_1i_2}_{i_2i_2} + v^{j_1j_2}_{j_1j_2} + w^{k_1k_2}_{k_1k_2}) = 0$. Since our choice of bases is arbitrary, we may assume that $T^{i_1j_1k_1} \neq 0 \neq S^{i_2j_2k_2}$. Then taking different values of k_1, k_2 and fixing i_1, i_2, j_1, j_2, we see all the $w^{k_1k_2}$ must be equal and similarly for U and V. This shows that $U = \lambda \mathrm{Id}_{A_1\otimes A_2}$, $V = \mu \mathrm{Id}_{B_1\otimes B_2}$ and $W = \mathrm{Id}_{C_1\otimes C_2}$. By evaluating (7) one last time, we see $\lambda = \mu = \nu$ that is $L \in \mathfrak{z}_{A_1\otimes A_2, B_1\otimes B_2, C_1\otimes C_2}$ and this concludes the proof.

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