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On minimal free resolutions of sub-permanents and other ideals arising in complexity theory

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ABSTRACT

We compute the linear strand of the minimal free resolution of the ideal generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix and of the ideal generated by square-free monomials of degree k. The latter calculation gives the full minimal free resolution by [1]. Our motivation is to lay groundwork for the use of commutative algebra in algebraic complexity theory. We also compute several Hilbert functions relevant for complexity theory.

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1. Introduction

We study homological properties of two families of ideals over polynomial rings: the ideals $\mathcal{I}^{sqf;n,k} \subset \mathbb{C}[x_1, \ldots, x_n]$ generated by square-free monomials of degree k in n variables and the ideals $\mathcal{I}^{perm_n,k} \subset \mathbb{C}[x_{i,j}]_{1\leq i,j\leq n}$ generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix. Recall that the permanent of an $m \times m$ matrix $Y = (y_{i,j})$ is the polynomial

$$perm_m(Y) = \sum_{\sigma\in\mathfrak{S}_m} y_{1,\sigma(1)}y_{2,\sigma(2)}\cdots y_{m,\sigma(m)},$$

where $\mathfrak{S}_m$ denotes the symmetric group on m elements.

We obtain our results via larger ideals $I_{1\times k}(1, n)$ (resp. $I_{1\times k}(n, n)$). The ideal $I_{1\times k}(1, n)$ is generated by all monomials of degree k in n variables. The ideal $I_{1\times k}(n, n)$ is generated by permanents of $k \times k$ matrices produced from X where repetition of rows and columns is allowed. Invariantly, $I_{1\times k}(1, n) = \oplus_{j\geq k}S^j\mathbb{C}^n$ is the ideal generated by $S^k\mathbb{C}^n$ and $I_{1\times k}(n, n) \subset Sym(\mathbb{C}^{n^2})$ is the ideal generated $S^k\mathbb{C}^n \otimes S^k\mathbb{C}^n \subset S^k(\mathbb{C}^n \otimes \mathbb{C}^n)$. The main result in each case says that the linear strand of resolution of $\mathcal{I}^{sqf;n,k}$ (resp. $\mathcal{I}^{perm_n,k}$) is the subcomplex of the linear strand of the resolution of $I_{1\times k}(1, n)$ (resp. of $I_{1\times k}(n, n)$) consisting of elements of *regular weights* (cf. §2).

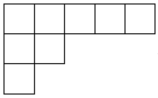
Our motivation comes from complexity theory. We seek to find differences between the homological behavior of ideals generated by $k \times k$ minors (i.e., subdeterminants) of the generic matrix and the ideals generated by $k \times k$ subpermanents. The ideal generated by square-free monomials arises as the $(n-k)$-th Jacobian ideal of the monomial $x_1x_2 \ldots x_n$.

2. Preliminaries

2.1. Representation theory

For proofs of the statements here, see, e.g., [3] or [9, Ch. 2]. We work exclusively over the complex numbers $\mathbb{C}$, although our results hold for an arbitrary field of characteristic 0. If W is a $\mathbb{C}$-vector space of dimension n, a choice of basis determines a maximal torus of diagonal matrices and a labeling of weights for the torus by n-tuples $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$. A weight λ is *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Irreducible representations of $GL(W)$ are in one-to-one correspondence with dominant weights λ. Let $S_\lambda W$ denote the irreducible representation associated to λ. Write $|\lambda| = \lambda_1 + \cdots + \lambda_n$ for the size of λ. A weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ is *regular* if each α_i is equal to 0 or 1. The regular weights will play an important role in stating our results.

When λ is a dominant weight with $\lambda_n \geq 0$, we say that λ is a *partition* of $r = |\lambda|$, and we write $\lambda \vdash r$. When dealing with partitions we often omit the trailing zeros. Associated to a partition is its *Young diagram* which consists of left–justified rows of boxes, with λ_i boxes in the i-th row: for example, the Young diagram associated to $\lambda = (5, 2, 1) \vdash 8$ is



The *transpose* λ' of a partition λ is obtained by transposing the corresponding Young diagram. For the example above, $\lambda' = (3, 2, 1, 1, 1)$.

Given finite dimensional $\mathbb{C}$-vector spaces F, G, the *Cauchy formulas* describe the decomposition of the symmetric and exterior powers of $F \otimes G$ into a sum of irreducible $GL(F) \times GL(G)$-representations, see, e.g., [9, Cor. 2.3.3]:

$$\begin{aligned} Sym^n(F \otimes G) &= \bigoplus_{\lambda \vdash n} S_\lambda F \otimes S_\lambda G, \\ \bigwedge^n (F \otimes G) &= \bigoplus_{\lambda \vdash n} S_\lambda F \otimes S_{\lambda'} G. \end{aligned} \tag{1}$$

Let $I_{a\times b}(m, n)$ denote the ideal generated by $S_{b^a}\mathbb{C}^m \otimes S_{b^a}\mathbb{C}^n$.

2.2. GL_m and $\mathfrak{S}_m$ representations

Let $E = \mathbb{C}^m$ equipped with its standard basis. The symmetric group $\mathfrak{S}_m$ is then contained in $GL(E)$ as the permutation matrices. Consider the irreducible representation $S_\lambda E$ where λ is a partition of m. Inside of $S_\lambda E$ we have the $\mathfrak{S}_m$-submodule spanned by the elements of regular weight $(1^m) = (1, 1, \ldots, 1)$. This submodule is denoted $[\lambda]$, the Specht module corresponding to λ. The representations $[\lambda]$ are the distinct irreducible representations of $\mathfrak{S}_m$ (see, e.g., [5]). Write

$$(S_\lambda \mathbb{C}^m)_{reg} = [\lambda].$$

Recall that for finite groups $H \subset G$, and an H-module W, $Ind_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is the induced G-module. For $n \geq m$ and $\lambda \vdash m$,

$$(S_\lambda \mathbb{C}^n)_{reg} \equiv Ind_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}([\lambda] \otimes [n-m]).$$

Introduce the notation $\tilde{\mathfrak{S}}_k = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n$, and if π is a partition of k, write $\widetilde{[\pi]} = [\pi] \times [n-k]$ for the $\tilde{\mathfrak{S}}_k$ module that is $[\pi]$ as an $\mathfrak{S}_k$-module and trivial as an $\mathfrak{S}_{n-k}$-module.

2.3. Finite free resolutions

Let $S = \mathbb{C}[x_1, \ldots, x_N]$ be the ring of polynomials in N variables equipped with its grading by degree. Let S_i denote its i-th graded component. Let S_+ denote the maximal ideal $S_+ = \oplus_{i>0} S_i$. Let $M = \oplus_{i\geq 0} M_i$ be a graded S-module. A complex of free graded S-modules

$$\mathbb{F} : 0 \to \mathbb{F}_N \stackrel{d_N}{\to} \mathbb{F}_{N-1} \to \cdots \to \mathbb{F}_1 \stackrel{d_1}{\to} \mathbb{F}_0$$

is a minimal free resolution of M if the only homology of $\mathbb{F}$ is $H_0(\mathbb{F}) = M$, and d_i are maps of degree 0 such that $d_i(\mathbb{F}_i) \subset S_+\mathbb{F}_{i-1}$ for all i.

Define the *graded Betti numbers* $\beta_{i,j}$ of M by

$$\mathbb{F}_i = \oplus_{j\geq 0} S(-i-j)^{\beta_{i,j}}$$

where $S(-m)$ denotes a copy of S with generator in degree m.

If M is an ideal generated in degree k, then $\mathbb{F}_i = \oplus_{j\geq k} S(-i-j)^{\beta_{i,j}}$. The linear strand of $\mathbb{F}$ is a subcomplex

$$\mathbb{F}^{lin} : 0 \to \mathbb{F}_N^{lin} \stackrel{d_N}{\to} \mathbb{F}_{N-1}^{lin} \to \cdots \to \mathbb{F}_1^{lin} \stackrel{d_1}{\to} \mathbb{F}_0^{lin}$$

where $\mathbb{F}_i^{lin} = S(-i-k)^{\beta_{i,i+k}}$. The graded Betti numbers $\beta_{i,j}$ have the interpretation in terms of Tor functors

$$\beta_{i,j} = \dim_{\mathbb{C}} Tor_i^S(S/S_+, M)_{i+j}$$

where the subscript denotes the homogeneous component. This means that $\beta_{i,j}$ can be calculated as

$$Tor_i^S(S/S_+, M)_{i+j} = H_i(\mathbb{C}(x_1, \ldots, x_N; M))_{i+j} \tag{2}$$

where $\mathbb{C}(x_1, \ldots, x_N; M)$ is the *Koszul complex* defined by $\mathbb{C}(x_1, \ldots, x_N; M)_i = \bigwedge^i \mathbb{C}^N \otimes_S M$ and the differential $d : \bigwedge^i \mathbb{C}^N \otimes_S M \to \bigwedge^{i-1} \mathbb{C}^N \otimes_S M$ by

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i} \otimes m) = \sum_{u=1}^{i} (-1)^{u+1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_u} \wedge \cdots \wedge e_{j_i} \otimes x_{j_u} m.$$

3. The linear strands of the minimal free resolutions of the ideals $\mathcal{I}^{perm_n,k}$

3.1. The resolution of $I_{1\times k}(n, n)$

The ideal $I_{1\times k}(n, n)$ is the ideal generated by $S_k E \otimes S_k F \subset S^k(E \otimes F)$, where $E, F \simeq \mathbb{C}^n$. The minimal free resolution of this ideal is known (see, e.g., [7], where it is denoted by $I_{1\times k}$). The linear components of this resolution are generated by

$$\underline{\mathbb{F}}_j^{lin} = \bigoplus_{a+b=j} S_{(k+b,1^a)} E. \tag{3}$$

So the j-th linear term is $\mathbb{F}_j^{lin} = \underline{\mathbb{F}}_j^{lin} \otimes S_{(k+a,1^b)} F \otimes S(-k-j)$.

Since the resolution is $GL(E) \times GL(F)$-equivariant, each module in the complex has a double weight decomposition induced by the restricted action of pairs of diagonal matrices.

3.2. The main result

We work over $S = \mathbb{C}[x_{i,j}]_{1\le i,j\le n} = Sym(E \otimes F)$.

Define a sub-complex $\mathbb{H}^{lin}$ of the complex $\mathbb{F}^{lin}$ given by (3) by setting $\underline{\mathbb{H}}_j^{lin}$ to be the subspace of $\underline{\mathbb{F}}_j^{lin}$ spanned by the basis elements of regular content. Note that $\mathbb{H}^{lin}$ is indeed a sub-complex of $\mathbb{F}^{lin}$. Let $E_j \subset E$, $F_j \subset F$ denote the span of the first j basis vectors.

Theorem 3.1. *When $k > 1$, the complex $\mathbb{H}^{lin}$ is the linear part of the minimal free resolution of the ideal $\mathcal{I}^{perm_n,k}$. Moreover,* $\dim \underline{\mathbb{H}}_j = \binom{n}{\kappa+j}^2 \binom{2(\kappa+j-1)}{j}$.

As an $\mathfrak{S}_n \times \mathfrak{S}_n$-module,

$$\underline{\mathbb{H}}_j^{lin} = Ind_{\tilde{\mathfrak{S}}_{\kappa+j}\times\tilde{\mathfrak{S}}_{\kappa+j}}^{\mathfrak{S}_n\times\mathfrak{S}_n}\Big(\bigoplus_{a+b=j} [\widetilde{\kappa+b}, 1^a]_{E_{\kappa+j}} \otimes [\widetilde{\kappa+a}, 1^b]_{F_{\kappa+j}}\Big). \tag{4}$$

Proof. Consider the complex $\mathbb{F}^{lin}$ giving the linear strand of the ideal $I_{1\times k}(n, n)$. The term $\mathbb{F}_0$ consists of the generators of $I_{1\times k}(n, n)$, the space $S_kE \otimes S_kF$.

Inside $S_kE \otimes S_kF$ is the ideal generated by the sub-permanents which consists of the subspace of regular weights. Note that the set of regular vectors in any $E^{\otimes m} \otimes F^{\otimes m}$ (where we assume $m \le n$ in order for the set of such vectors to be nonempty) spans a $\mathfrak{S}_E \times \mathfrak{S}_F$-submodule.

The linear strand of the j-the term in the minimal free resolution of the ideal $\mathcal{I}^{perm_n,k}$ is also a $\mathfrak{S}_E \times \mathfrak{S}_F$-submodule of $\mathbb{F}_j$. We claim this sub-module is generated by the span of the regular vectors. In what follows $p(i_1, \dots, i_k; j_1, \dots, j_k)$ denotes the sub-permanent formed from rows $i_1, \dots i_k$ and columns $j_1, \dots, j_k$.

We work by induction, the case $j = 0$ was discussed above. Assume the result has been proven up to homological degree $j - 1$ and consider the homological degree j and homogeneous degree $k + j$. The generators of the j-th module in the linear strand of the resolution of $\mathcal{I}^{perm_n,k}$ have to be contained in linear part of $\mathbb{H}_{j-1}^{lin}$, so all its weights are either regular, or such that one of the row indices i_α is 2, and/or one of the column indices j_β is 2, and all other p_u, q_u are zero or 1. Call such a weight *sub-regular*. It remains to show that no linear syzygy with a sub-regular weight can appear. To do this we show that no sub-regular weight vector in $(\mathbb{F}_j)_{subreg}$ maps to zero in $(\mathbb{H}_{j-1}) \cdot (E \otimes F)$.

First consider the case where both the E and F weights are sub-regular, then (because the space is a $\mathfrak{S}_E \times \mathfrak{S}_F$-module), the weight $(2, 1, \dots, 1, 0, \dots, 0) \times (2, 1, \dots, 1, 0, \dots, 0)$ must appear in the syzygy. The only way for this to appear is to have a term divisible by $x_{1,1}$. But, since $x_{1,1}$ is not a zero-divisor in $Sym(V)$, such a term cannot map to zero because our syzygy is a syzygy of degree zero multiplied by $x_{1,1}$. But by minimality no such syzygy exists.

Finally consider the case where there is a vector of weight $(2, 1^{j+k-2}) \times (1^{j+k})$ appearing. Here it is more convenient to look at the calculation of the free resolution using the Koszul complex. Such a syzygy would give a Koszul cycle with summands of the form

$$z = \sum_t a_t e_{a_{1,t},b_{1,t}} \wedge \cdots \wedge e_{a_{j,t},b_{j,t}} \otimes p(I_t; J_t) \tag{5}$$

where $p(I_t; J_t)$ are subpermanents formed from distinct rows and columns and $a_t \in \mathbb{C}$. The total weight is $(2, 1^{k+j-2}) \times (1^{k+j})$ and the Koszul differential $d(z)$ is zero. Consider this differential. The coefficients of all the basis elements $e_{a_1,b_1} \wedge \cdots \wedge e_{a_{j-1},b_{j-1}}$ of $d(z)$ have to be zero. There are three kinds of basis elements: the indices $a_1, \ldots, a_{j-1}$ can contain number 1 twice, once or not contain 1 at all. Consider the basis elements not containing 1, say the element $e_{2,1} \wedge \cdots \wedge e_{j,j-1}$. The only elements that can appear on the right hand side of the tensor product in $d(z)$ are the elements $x_{1,s}p(1, j+1, j+2, j+k-1; j, j+1, j+2, \ldots, \hat{s}, \cdots, j+k)$, for $s = j, j+1, \ldots, j+k$.

Lemma 3.2. *Let $k > 1$. The elements $x_{1,s}p(1, j+1, j+2, j+k-1; j, j+1, j+2, \ldots, \hat{s}, \cdots, j+k)$, for $s = j, j+1, \ldots, j+k$ are linearly independent in S.*

Proof. After re-labeling, the lemma amounts to showing the polynomials $x_{1,1}P_1, \ldots, x_{1,k+1}P_{k+1}$ are linearly independent, where P_i is the permanent of the matrix obtained by removing the i-th column of

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k+1} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{k,1} & x_{k,2} & \cdots & x_{k,k+1} \end{pmatrix}.$$

Say that

$$\sum_{s=1}^{k+1} b_s x_{1,s} P_s = 0 \tag{6}$$

for some scalars b_s. We need to show that all b_s are zero. By symmetry it suffices to show that $b_1 = b_2 = b_3 = 0$. So set $x_{1,s} = 0$ for $s > 3$. Using the Laplace expansion of permanents along the first row, and writing $P_{i,j}$ for the permanent obtained by removing row 1 and columns i, j we can rewrite (6) as

$$b_1 x_{1,1}(x_{1,2}P_{1,2} + x_{1,3}P_{1,3}) + b_2 x_{1,2}(x_{1,1}P_{1,2} + x_{1,3}P_{1,3}) + b_3(x_{1,1}P_{1,3} + x_{1,2}P_{2,3}) = 0$$

which gives $b_i + b_j = 0$ for $1 \le i < j \le 3$, so $b_1 = b_2 = b_3 = 0$. $\square$

Lemma 3.2 implies that in all the summands in z of (5) with $a_t \neq 0$, both appearances of the index 1 have to occur among $a_{1,t}, \cdots, a_{j,t}$. Now consider the coefficient of the basis element where 1 occurs among $a_1, \ldots, a_{j-1}$, say, $e_{1,1} \wedge e_{2,2} \wedge \cdots \wedge e_{j-1,j-1}$ in $d(z)$. We obtain a linear combination of the elements $x_{1,s}p(j, j+1, \cdots, j+k-1; j, j+1, \cdots, \hat{s}, \cdots, j+k)$ for $s = j, j+1, \ldots, j+k$. But these elements are trivially linearly independent (all monomials occurring in them are different) so all coefficients a_t are zero.

The rest of Theorem 3.1 follows because if π is a partition of $\kappa + j$ then the weight $(1,\ldots,1)$ subspace of $S_\pi E_{\kappa+j}$, considered as an $\mathfrak{S}_{E_{\kappa+j}}$-module, is $[\pi]$ (see, e.g., [5]), and the space of regular vectors in $S_\pi E \otimes S_\mu F$ is $Ind_{\tilde{\mathfrak{S}}_{E_{\kappa+j}} \times \tilde{\mathfrak{S}}_{F_{\kappa+j}}}^{\mathfrak{S}_E \times \mathfrak{S}_F} \widetilde{[\pi]}_E \otimes \widetilde{[\mu]}_F$. In the formula for the dimension of $\underline{\mathbb{H}}_j$, the factor $\binom{n}{k+j}^2$ is explained by inducing. The dimensions of hook Specht modules are binomial coefficients, dim $[x, 1^y] = \binom{x+y-1}{y}$ So we need to prove that

$$\sum_{a+b=j} \binom{k+j-1}{a}\binom{k+j-1}{b} = \binom{2(k+j-1)}{j}$$

This has a combinatorial explanation. Given $2(k+j-1)$ balls, $k+j-1$ white and $k+j-1$ black, both sides of the equation calculate number of choices of $k+j-1$ of them: the left side partitions into how many white (a) and how many black (b) are chosen. □

Remark 3.3. For small n and κ, computer computations show no additional first syzygies on the $\kappa \times \kappa$ sub-permanents of a generic $n \times n$ matrix (besides the linear syzygies) in degree less than the degree of the Koszul relations 2κ. For example, for $\kappa = 3$ and $n = 5$, there are 100 cubic generators for the ideal and 5200 minimal first syzygies of degree six. There can be at most $\binom{100}{2} = 4950$ Koszul syzygies, so there must be additional non-Koszul first syzygies.

4. The minimal free resolutions of the ideals $\mathcal{I}^{sqf;n,k}$

4.1. The resolutions of $I_{1\times k}(1, n)$

Next we consider the case $E = \mathbb{C}$, $F = \mathbb{C}^n$. In this case $S = Sym(F)$ and the ideal $I_{1\times k}(1, n)$ is just the ideal generated by all monomials of degree k. The resolution of this ideal is well-known, see, e.g., [7] or [2].

The whole resolution is linear and $GL(F)$-equivariant, and its k-th term is

$$\mathbb{F}_j = S_{(k,1^j)} F \otimes S(-k-j). \tag{7}$$

4.2. The resolution of $\mathcal{I}^{sqf;n,k}$

We work over $S = \mathbb{C}[x_1, \ldots, x_n] = Sym(F)$.

Define a subcomplex $\mathbb{H}^{lin}$ of the complex $\mathbb{F}^{lin}$ given by (7) by setting $\underline{\mathbb{H}}_j^{lin}$ to be the subspace of $\underline{\mathbb{F}}_j^{lin}$ spanned by the basis elements of regular weight. Note that

$$\mathbb{H}^{lin} = \oplus_j \underline{\mathbb{H}}_j^{lin} \otimes S(-k-j)$$

is indeed a subcomplex of $\mathbb{F}^{lin}$.

Theorem 4.1. *The complex $\mathbb{H}^{lin}$ is the linear part of the minimal free resolution of the ideal $\mathcal{I}^{sqf;n,k}$. We have the $\mathfrak{S}_n$-module decomposition*

$$\underline{\mathbb{H}}_j^{lin} = Ind_{\tilde{\mathfrak{S}}_{\kappa+j}}^{\mathfrak{S}_n} \widetilde{[\kappa, 1^j]}, \tag{8}$$

which has dimension $\binom{\kappa+j-2}{j-1}\binom{n}{\kappa+j}$.

Proof. We want to show that $\mathbb{H}^{lin}$ is the linear strand of the resolution of $\mathcal{I}^{sqf;n,k}$. We proceed by induction on j. The case $j = 0$ is clear because the generators of $\mathcal{I}^{sqf;n,k}$ are precisely the generators of $I_{1\times k}(1, n)$ with regular weights. Assume we proved the result for $j - 1$ and consider the j-th module in the resolution. As with the subpermanent case, it is enough to consider the elements of a subregular weight as linear relations between elements of regular weight are either of regular or subregular weight.

Consider the syzygies in homological dimension j and in homogeneous degree $k + j$ in term of cycles in the Koszul complex $\mathbb{C}(x_1, \dots, x_n; \mathcal{I}^{sqf;n,k})$. These will be cycles of the form

$$z = \sum_t a_t e_{a_{1,t}} \wedge \dots \wedge e_{a_{j,t}} \otimes x_{u_{1,t}} x_{u_{2,t}} \dots x_{u_{k,t}}$$

where the total weight is $(2, 1^{k+j-2})$ and all monomials $x_{u_{1,t}} x_{u_{2,t}} \dots x_{u_{k,t}}$ are of regular weights, and a_t are scalars. In each summand with $a_t \neq 0$ we are forced to have one 1 among $a_{i,t}$ and one 1 among $u_{i,t}$. So we can assume that in each summand with $a_t \neq 0$ we have $a_{1,t} = 1$ and $u_{1,t} = 1$. We have $d(z) = 0$. But, looking at the coefficient of $d(z)$ with respect to the basis vector $e_{a_{2,t}} \wedge \dots \wedge e_{a_{j,t}}$ we see that its coefficient is just $a_t x_1 x_{u_{1,t}} x_{u_{2,t}} \dots x_{u_{k,t}}$ which forces a_t to be zero. The dimension formula follows as in Theorem 3.1. □

Remark 4.2. The easiest way to see that the resolution of the ideal $\mathcal{I}^{sqf;n,k}$ is linear and to see the ranks of the modules is to observe that for the $k \times n$ matrix

$$M(A, X) = \begin{pmatrix} a_{1,1}x_1 & \dots & a_{1,n}x_n \\ \dots & \dots & \dots \\ a_{k,1}x_1 & \dots & a_{k,n}x_n \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ a_{k,1} & \dots & a_{k,n} \end{pmatrix}$$

is a matrix of scalars with all maximal minors non-zero, the ideal of maximal minors of the matrix $M(A, X)$ is just $\mathcal{I}^{sqf;n,k}$, so the resolution in question is an Eagon–Northcott complex ([2] or [9, 6.1.6]). After this paper was submitted, a characteristic free description of the resolution of $\mathcal{I}^{sqf;n,k}$, with explicit differentials appeared in [4].

5. Additional results

For $P \in S^d\mathbb{C}^N$, let $I^{P,k} \subset S^k\mathbb{C}^N$ denote the ideal generated by the partial derivatives of P of order $d-k$.

5.1. Size two subpermanents

Theorem 5.1. *Let $I_t^{\mathrm{perm}_n,2}$ denote the degree t component of the ideal generated by the size two sub-permanents of an $n \times n$ matrix, so $\dim I_2^{\mathrm{perm}_n,2} = \binom{n}{2}^2$. Then*

$$\dim \mathbb{C}[x_{i,j}]/I_t^{perm_n,2} = \binom{n^2+t-1}{t} - [\binom{n}{t}^2 + n^2 + (t-1)(\binom{n^2}{2} - \binom{n}{2}^2) + 2\binom{t-1}{2}(\binom{n}{2}^2 + n\binom{n}{3}) + 2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}],$$

where recall that $\binom{n}{t} = 0$ for $t > n$, in which case the formula is $\dim S^t\mathbb{C}^{n^2}$ minus the value of the Hilbert polynomial at t.

First, the Hilbert polynomial:

Theorem 5.2. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n \times n$ matrix, the Hilbert polynomial of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is*

$$\sum_{i=0}^{n} f_i \binom{t-1}{i}, \tag{9}$$

where f_i is the i-th entry in the vector

$$\left[n^2, \binom{n^2}{2} - \binom{n}{2}^2, 2\binom{n}{2}^2 + 2n\binom{n}{3}, 2n\binom{n}{4}, 2n\binom{n}{5}, \ldots, 2n\binom{n}{n}\right].$$

Proof. [6, Thm. 3.2] gives a Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$, the radical of $\mathcal{I}^{\mathrm{perm}_n,2}$, and by [6, Thm. 3.3], $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ has finite length, so vanishes in high degree. The Hilbert polynomial only measures dimension asymptotically, so

$$HP(Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}, t) = HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t).$$

By [6], for any diagonal term order, the Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is given by quadrics of the form

$$x_{ij}x_{kl} + x_{kj}x_{il} \quad \text{with } i<k,\ j<l,$$

and five sets of cubic monomials

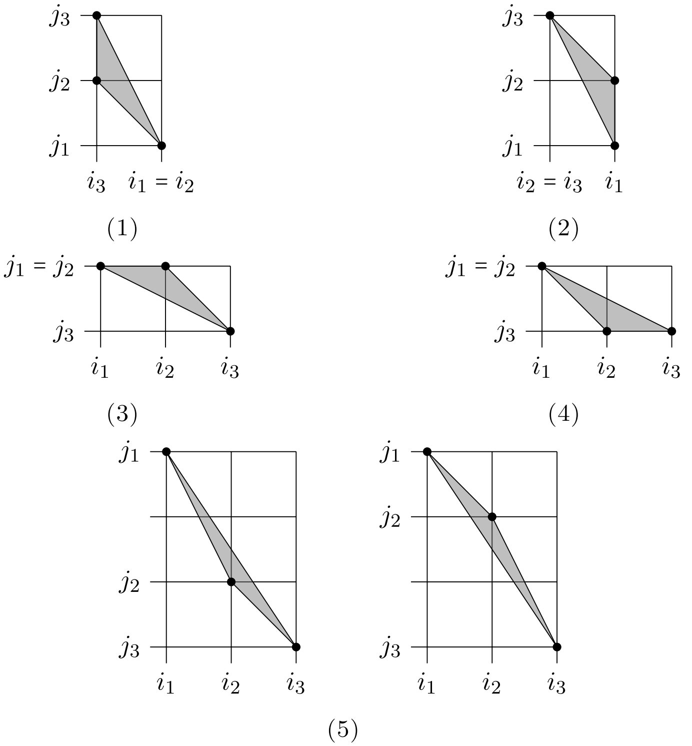


Fig. 1. Non-triangles of Δ from Equation (10).

$$\begin{array}{lcc} x_{i_1j_1}x_{i_1j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_1}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_3} & i_1 < i_2 < i_3 & j_1 > j_2 > j_3. \end{array} \tag{10}$$

The key observation is that all the cubic monomials are square-free, as are the initial terms of the quadrics. Thus the initial ideal of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is a square-free monomial ideal and corresponds to the Stanley–Reisner-ideal of a simplicial complex Δ. By [8, Lemma 5.2.5], the Hilbert polynomial is as in Equation (9), where f_i is the number of i-dimensional faces of Δ. As the vertex set of Δ corresponds to all lattice points (i,j) with $1 \le i,j \le n$, it is immediate that $f_0 = n^2$.

Since $x_{ij}x_{kl}$ is a non-face if $i < k$, $j < l$, no edge connects a southwest lattice point to a northeast lattice point. Hence, the edges of Δ consist of all pairs $(i,j),(k,l)$ with $i \ge k$ and $j \ge l$, of which there are $\binom{n^2}{2} - \binom{n}{2}^2$.

Next, consider the triangles of Δ. Equation (10) says there are no triangles in Δ of the types in Fig. 1. Also, there are no triangles which contain an edge connecting vertices at positions (i,j) and (k,l) with $i < k$, $j < l$. Thus, the only triangles in Δ are right triangles, but with hypotenuse sloping from northwest to southeast. For a lattice point v at position (d, e) there are exactly $(d-1)(e-1)$ right triangles having v as their unique north-most vertex. In the rightmost column n, there are no such triangles, in the next to last column $n-1$ there are $(n-1)+(n-2)+\cdots = \binom{n}{2}$ such triangles. Continuing this way yields a total count of

$$(n-1)\binom{n}{2} + (n-2)\binom{n}{2} + \cdots 2\binom{n}{2} + \binom{n}{2} = \binom{n}{2}^2$$

such right triangles, and taking into account the right triangles for which v is the unique south-most vertex doubles this number.

However, this count neglects thin triangles–those which have all vertices in the same row or column. Since the number of thin triangles is $2n\binom{n}{3}$, the final count for the triangles of Δ is

$$2\binom{n}{2}^2 + 2n\binom{n}{3}.$$

For tetrahedra, the conditions of Equation (10) imply that there can only be thin tetrahedra, and an easy count gives $2n\binom{n}{4}$ such. The same holds for higher dimensional simplices, and concludes the proof. □

Corollary 5.3. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n\times n$ matrix, the Hilbert function of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is, when $t \geq 3$,*

$$HF(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t) = \binom{n}{t}^2 + HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t), \tag{11}$$

and it equals the Hilbert polynomial for $t > n$.

Proof. The Hilbert function of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ in degree t is $\binom{n}{t}^2$ by [6, Thm. 3.3]. The result follows by combining Equation 9 with the short exact sequence

$$0 \longrightarrow \sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2} \longrightarrow Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2} \longrightarrow Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}} \longrightarrow 0,$$

and additivity of the Hilbert function. □

For the purposes of comparing with other ideals, we rephrase this as:

Corollary 5.4. $\dim I_2^{\mathrm{perm}_n,2} = \binom{n}{2}^2$. *For $t \geq 3$:*

$$\begin{aligned}\dim I_t^{perm_n,2} = &\binom{n^2+t-1}{t} - [\binom{n}{t}^2 + n^2 + (t-1)(\binom{n^2}{2} - \binom{n}{2}^2) \\ &+ 2\binom{t-1}{2}(\binom{n}{2}^2 + n\binom{n}{3}) + 2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}].\end{aligned}$$

5.2. Hilbert functions for ideals of square-free monomials

Although these can be deduced from our resolutions, we present the Hilbert functions and polynomials for the ideals generated by square-free monomials.

Proposition 5.5. *The Hilbert function of $\mathcal{I}^{x_1\cdots x_n,\kappa}$ in degree $\kappa+t$ is*

$$\dim \mathcal{I}^{(x_1\cdots x_n),\kappa}_{\kappa+t} = \sum_{j=0}^{n-\kappa} \binom{n}{\kappa-j}\binom{\kappa+t-1}{\kappa+j-1} \tag{12}$$

Proof. The ideal in degree $d = t+\kappa$ has a basis of the distinct monomials of degree d containing at least $n-k$ distinct indices. When we divide such a basis vector by $x_1\cdots x_n$ the denominator will have degree at most κ. For each $i \leq \kappa$, the space of possible numerators with a denominator of degree i that is fixed, has dimension $\dim S^{d-n+i}\mathbb{C}^{n-i}$, and there are $\binom{n}{i}$ possible denominators. Summing over i gives the result. □

For the Hilbert function of the coordinate ring, we have the following expression:

Proposition 5.6. *The Hilbert function of $Sym(\mathbb{C}^n)/\mathcal{I}^{x_1\cdots x_n,\kappa}$ in degree t is*

$$\dim(Sym(\mathbb{C}^n)/\mathcal{I}^{x_1\cdots x_n,\kappa})_t = \sum_{j=0}^{n-\kappa-2}\binom{n}{j+1}\binom{t-1}{j}, \tag{13}$$

if $t \geq n-\kappa-1$, and $\binom{n+t-1}{n-1}$ if $t < n-\kappa-1$.

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