ABELIAN TENSORS

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Abstract. We analyze tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfying Strassen's equations for border rank m. Results include: two purely geometric characterizations of the Coppersmith-Winograd tensor, a reduction to the study of symmetric tensors under a mild genericity hypothesis, and numerous additional equations and examples. This study is closely connected to the study of the variety of m-dimensional abelian subspaces of $\mathrm{End}(\mathbb{C}^m)$ and the subvariety consisting of the Zariski closure of the variety of maximal tori, called the variety of reductions.

Sommaire. Nous étudions des tenseurs dans $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfaisant les équations de Strassen lorsque le rang du bord vaut m. Les résultats obtenus comprennent : deux caractérisations purement géométriques du tenseur de Coppersmith-Winograd, une réduction à l'étude des tenseurs symétriques sous une hypothèse raisonnable de généricité, et beaucoup de nouveaux exemples et équations. Cette étude est liée de près à l'étude de la variété des sous-espaces abéliens de dimension m de $\mathrm{End}(\mathbb{C}^m)$ et la sous-variété obtenue comme l'adhérence de Zariski de la variété des tores maximaux, appelée variété des réductions.

1. Introduction

The rank and border rank of a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (defined below) are basic measures of its complexity. Central problems are to develop techniques to determine them (see, e.g., [27, 13, 15, 23]). Complete resolutions of these problems are currently out of reach. For example, neither problem is solved already in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. This article focuses on a very special class of tensors, those satisfying Strassen's commutativity equations (see §2.1). The study of such tensors is related to the classical problem of studying spaces of commuting matrices, see, e.g. [21, 44, 22, 26].

To completely understand border rank, it would be sufficient to understand the case of border rank m in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (see [27, Cor. 7.4.1.2]). We study this problem under two genericity hypotheses - *concision*, which essentially says we restrict to tensors that are not contained in some $\mathbb{C}^{m-1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, and 1_A-*genericity*, which is defined below. Even under these genericity hypotheses, the problem is still subtle.

Let A, B, C be complex vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, let $T \in A \otimes B \otimes C$ be a tensor. (In bases T is a three dimensional matrix of size $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.) We may view T as a linear map $T : A^* \to B \otimes C \simeq \mathrm{Hom}(C^*, B)$. (In bases, $T((\alpha_1, \cdots, \alpha_{\mathbf{a}}))$ is the $\mathbf{b} \times \mathbf{c}$ matrix α_1 times the first slice of the $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ matrix, plus α_2 times the second slice ... plus $\alpha_{\mathbf{a}}$ times the $\mathbf{a}$-th slice.) One may recover T up to isomorphism from the space of linear maps $T(A^*)$.

One says T has *rank one* if $T = a \otimes b \otimes c$ for some $a \in A$, $b \in B$ and $c \in C$, and the *rank* of T, denoted $\mathbf{R}(T)$ is the smallest r such that T may be expressed as the sum of r rank one tensors. Rank is not semi-continuous, so one defines the *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, to be the smallest r such that T is a limit of tensors of rank r, or equivalently (see e.g. [27, Cor. 5.1.1.5]) the smallest r such that T lies in the Zariski closure of the set of tensors of rank r.

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Write $\hat{\sigma}_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subset A \otimes B \otimes C$ for the variety of tensors of border rank at most r (the cone over the r-th secant variety of the Segre variety). We will be mostly concerned with the case $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$.

To make the connection with spaces of commuting matrices, we need to have linear maps from a vector space to itself. Define $T \in A \otimes B \otimes C$ to be 1_A*-generic* if there exists $\alpha \in A^*$ with $T(\alpha)$ invertible. Then $T(A^*)T(\alpha)^{-1} \subset \mathrm{End}(B)$ will be our space of endomorphisms and Strassen's equations for border rank m is that this space is *abelian*, i.e., in bases we obtain a space of commuting matrices.

Of particular interest is when an m-dimensional space of commuting matrices, viewed as a point of the Grassmannian $G(m, \mathrm{End}(B))$, is in the closure of the space of diagonalizable subspaces (i.e., the maximal tori in $\mathfrak{gl}_n$), which is denoted $Red(m)$ in [24]. Much of this paper will utilize the interplay between the tensor and endomorphism perspectives.

Our primary motivation for this paper comes from the study of the complexity of the matrix multiplication tensor $M_{\langle \mathbf{n} \rangle} \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$. We initiate a geometric study of the tensors used to prove upper bounds on the exponent of matrix multiplication, especially the Coppersmith-Winograd tensor. In [2] they showed that one cannot prove the exponent of matrix multiplication is less than exponent 2.3 using the laser method applied to the Coppersmith-Winograd tensor that was used for the current world record in [46, 39, 32]. The authors suggested that to improve the upper bound on the exponent one should look for other tensors that give even better upper bounds via the laser method. While the tensors of Strassen and Coppersmith-Winograd were defined in terms of their *combinatorial* properties, we thought it would be useful to isolate their *geometric* properties, and use these geometric properties as a basis for the search. One geometric property is that they have (near) minimal border rank and relatively large rank. In this paper we find other tensors with the same property and we hope to investigate their *value* (in the sense of [46, 39, 32, 2]) in future work. On the other hand, to our surprise, *we isolate two further geometric properties that essentially characterize the Coppersmith-Winograd tensors*, see Theorems 7.4 and 7.5, which hints that one might have already reached the limits of the laser method.

Another motivation from computer science is the construction of *explicit* tensors of high rank and border rank, see, e.g., [1, 37]. We give several such examples.

Our results include

- Two purely geometric characterizations of the Coppersmith-Winograd tensor (Theorems 7.4 and 7.5).
- Determination of the ranks of numerous tensors of minimal border rank including: all 1_A-generic tensors that satisfy Strassen's equations for $m = 4$ and $m = 5$.
- Proof, in §6, when $m \leq 4$, of a conjecture of J. Rhodes [4, Conjecture 0] for tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ that their ranks cannot be twice their border ranks, and counter-examples for all $m > 4$ (Proposition 6.6).
- Explicit examples of tensors with rank to border rank ratio greater than two (Proposition 6.11 and Theorem 6.12).
- Proof that the flag algebras of [24] are of minimal border rank, §6.2.
- Explicit examples showing that the known necessary conditions for minimal border rank are independent.
- 1-generic tensors satisfying Strassen's equations, but far from minimal border rank, §5.
- Proof that 1-generic tensors satisfying Strassen's equations must be symmetric (Proposition 5.8).
- New necessary conditions for border rank to be minimal (Theorem 2.4) with an example (Example 5.6), answering a question of A. Leitner [33].

- A class of tensors for which Strassen's additivity conjecture holds (Theorem 4.1).

1.1. **Background and previous work.** The maximum rank of $T\in\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ is not known, it is easily seen to be at most m^2 (and known to be at most $\frac{2}{3}m^2$ [7]), and of course is at least the maximum border rank. The maximum border rank is $\lceil\frac{m^3-1}{3m-2}\rceil$ except when $m=3$ when it is five [34, 40]. In computer science, there is interest in producing explicit tensors of high rank and border rank. The maximal rank of a known explicit tensor is $3m-\log_2(m)-3$ when m is a power of two [1], see Example 3.3.

Tensors in $A\otimes B\otimes C$ are completely understood when all vector spaces have dimension at most three [11]. In particular, for tensors of border rank three, the maximum rank is five. The case of $\mathbb{C}^2\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ is also completely understood, see, e.g. [27, §10.3]. While tensors of border rank 4 in $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ are essentially understood [19, 20, 5], the ranks of such tensors are not known. We determine their ranks under our two genericity hypotheses. The difficulty of understanding border rank four tensors in $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ (which was first overcome in [19]) was non-concision, which we avoid in this paper.

Let $Red^{0,SL}(m)$ denote the set of all maximal tori in $SL(m)$, i.e., the set of all $(m-1)$-dimensional abelian subgroups that are diagonalizable. It can be given a topology (called the *Chabauty topology*, see [33]) and its closure $Red^{SL}(m)$ is studied in [33]. (A. Leitner works over $\mathbb{R}$, but this changes little.) If one considers the corresponding Lie algebras, one obtains a subvariety of the Grassmannian $Red^{\mathfrak{sl}}(m)=Red(m)\subset G(m-1,\mathfrak{sl}_m)$ that was studied classically, and is called the *variety of reductions* in [24]. More precisely,

$$Red(m)=\overline{\{E\in G(m-1,\mathfrak{sl}_m)\mid \text{a basis of } E \text{ is simultaneously diagonalizable}\}}.$$

One can equivalently prove results at the Lie group or Lie algebra level. As the Cartan subalgebras of $\mathfrak{sl}_m$ and $\mathfrak{gl}_m$ can be identified, by adding and dividing out by identity, we may equivalently work in the Grassmannian $G(m,\mathfrak{gl}_m)$.

We present the results of [33] (some of which we had found independently) in tensor language for the benefit of the tensor community.

A tensor $T\in A\otimes B\otimes C$ is A-*concise* if the map $T:A^*\to B\otimes C$ is injective, and it is *concise* if it is A, B and C concise. Equivalently, T is A-concise if it does not lie in any $A'\otimes B\otimes C$ with $A'\subsetneq A$. Note that if T is A-concise, then $\underline{\mathbf{R}}(T)\geq\mathbf{a}$.

Definition 1.1. If $\mathbf{b}=\mathbf{c}=m$ define $T\in A\otimes B\otimes C$ to be 1_A-*generic* if $T(A^*)$ contains an element of rank m. Define $1_B,1_C$ genericity similarly and say T is 1-*generic* if it is $1_A,1_B$ and 1_C-generic.

Note that if T is 1_A-generic, then T is B and C concise and in particular, $\underline{\mathbf{R}}(T)\geq m$.

1.2. **Organization.** In §2 we describe necessary conditions for 1_A-generic tensors to have border rank m. In addition to Strassen's equations, there is an End-closed condition, flag genericity conditions, and infinitesimal flag genericity conditions, the last of which is new. In §3, we describe the method of [1] for proving lower bounds on the ranks of explicit tensors. This method has a consequence for the study of Strassen's additivity conjecture that we describe in §4. In §5 we study 1_A-generic tensors satisfying Strassen's equations that have border rank greater than m, giving explicit examples where each of the necessary conditions fail and showing that such tensors can have very large border rank. Moreover, we show that a 1-generic tensor satisfying Strassen's equations is isomorphic to a symmetric tensor. In §6 we study 1_A-generic tensors of minimal border rank, presenting a sufficient condition to have minimal border rank, classifications when $m=4,5$, computing the ranks as well, and explicit examples of tensors with large gaps between rank and border rank. We conclude in §7 with a geometric analysis of tensors that have been useful for proving upper bounds on the complexity of the matrix multiplication tensor, in particular, giving two geometric characterizations of the Coppersmith-Winograd tensor.

1.3. **Notation.** Let V be a complex vector space, $V^* = \{\alpha : V \to \mathbb{C} \mid \alpha \text{ is linear}\}$ denotes the dual vector space, $V^{\otimes k}$ denotes the k-th tensor power, S^kV denotes the symmetric tensors in $V^{\otimes k}$, equivalently, the homogeneous polynomials of degree k on V^*, and Λ^kV denotes the skew-symmetric tensors in $V^{\otimes k}$. If $U \subset V$ (resp. $v \in V$), we let $U^\perp \subset V^*$ (resp. $v^\perp \subset V^*$) denote its annhilator.

Projective space is $\mathbb{P}V = (V \backslash 0)/\mathbb{C}^*$. For $v \in V$, $[v] \in \mathbb{P}V$ denotes the corresponding point in projective space and for any subset $Z \subset \mathbb{P}V$, $\hat{Z} \subset V$ is the corresponding cone in V. For a variety $X \subset \mathbb{P}V$, X_{smooth} denotes its smooth points. For $x \in X_{smooth}$, $\hat{T}_xX \subset V$ denotes its affine tangent space. For a subset $Z \subset V$ or $Z \subset \mathbb{P}V$, its Zariski closure is denoted $\overline{Z}$.

The irreducible polynomial representations of $GL(V)$ are indexed by partitions $\pi = (p_1, \cdots, p_q)$ with at most $\dim V$ parts. Let $\ell(\pi)$ denote the number of parts of π (so $\ell((p_1, \cdots, p_q)) = q$), and let $S_\pi V$ denote the irreducible $GL(V)$-module corresponding to π. The conjugate partition to π is denoted π'.

Since we lack systematic methods to prove bounds on the ranks of tensors, we often rely on the presentation of a tensor in a given basis to help us. For example, the structure tensor for the group algebra of $\mathbb{Z}_m$ in the standard basis looks like

$$M_{\mathbb{C}[\mathbb{Z}_m]}(A^*) = \{ \begin{pmatrix} x_0 & x_1 & \cdots & x_{m-1} \\ x_{m-1} & x_0 & x_1 & \cdots \\ \vdots & & \ddots & \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \mid x_j \in \mathbb{C} \}$$

but after a change of basis (the discrete Fourier transform), it becomes diagonalized so in the new basis it is transparently of rank and border rank m.

For $T \in A \otimes B \otimes C$, introduce the notation for $T(A^*)$ omitting the $x_j \in \mathbb{C}$, e.g., for $M_{\mathbb{C}[\mathbb{Z}_m]}(A^*)$, we write

$$M_{\mathbb{C}[\mathbb{Z}_m]}(A^*) = \begin{pmatrix} x_0 & x_1 & \cdots & x_{m-1} \\ x_{m-1} & x_0 & x_1 & \cdots \\ \vdots & & \ddots & \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix}. \tag{1}$$

For tensors $T, T' \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A \otimes B \otimes C$, we will say T and T' are *strictly isomorphic* if there exists $g \in GL(A) \times GL(B) \times GL(C)$ such that $g(T) = T'$, and we will say T, T' are *isomorphic* if there exists $g \in GL(A) \times GL(B) \times GL(C)$ and $\sigma \in \mathfrak{S}_3$ such that $\sigma(g(T)) = T'$.

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2. Border rank m equations for tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$

2.1. **Strassen's commutativity equations.** Throughout this sub-section $\dim B = \dim C = m$. Given $T \in A \otimes B \otimes C$ and $\alpha \in A^*$, consider $T(\alpha) \in B \otimes C = \mathrm{Hom}(C^*, B)$. If $T(\alpha)$ is invertible, for all $\alpha' \in A^*$, we may consider $T(\alpha')T(\alpha)^{-1} : B \to B$. Let $[X, Y] = XY - YX$ denote the commutator of the matrices X, Y.

Strassen's equations [40] are: for all $\alpha, \alpha_1, \alpha_2 \in A^*$ with $T(\alpha)$ invertible,,

$$\mathrm{rank}[T(\alpha_1)T(\alpha)^{-1}, T(\alpha_2)T(\alpha)^{-1}] \leq 2(\underline{\mathbf{R}}(T) - m).$$

In particular, if $\underline{\mathbf{R}}(T) = m$, then the space $T(A^*)T(\alpha)^{-1} \subset \mathrm{End}(B)$ is abelian. It is also useful to use Ottaviani's formulation of Strassen's equations [35]: consider the map

$$T_A^{\wedge} : B^* \otimes A \to \Lambda^2 A \otimes C$$
$$\beta \otimes a \mapsto a \wedge T(\beta).$$

If $\dim A = 3$, $\underline{\mathbf{R}}(T) \geq \frac{1}{2}\mathrm{rank}(T_A^{\wedge})$. If one restricts T to a 3-dimensional subspace of A^*, the same conclusion holds. In general $\mathrm{rank}(T_A^{\wedge}) \leq (\mathbf{a} - 1)\underline{\mathbf{R}}(T)$, because for a rank one tensor $a \otimes b \otimes c$, $(a \otimes b \otimes c)_A^{\wedge}(A \otimes B^*) = a \wedge A \otimes c$, i.e. $\mathrm{rank}(a \otimes b \otimes c)_A^{\wedge} = \mathbf{a} - 1$.

To deal with the case where $T(\alpha)$ is not invertible in Strassen's formulation, recall that a linear map $f : B \to C^*$ induces linear maps $f^{\wedge k} : \Lambda^k B \to \Lambda^k C^*$, and that $\Lambda^{m-1} B \simeq B^* \otimes \Lambda^m B$. Thus $f^{\wedge(m-1)} : \Lambda^{m-1} B \to \Lambda^{m-1} C^*$ may be identified with (up to a fixed choice of scale) a linear map $B^* \to C$, and thus its transpose may be identified with a linear map $C^* \to B$. If f is invertible, this linear map coincides up to scale with the inverse. In bases it is given by the cofactor matrix of f. So to obtain polynomials, use $(T(\alpha)^{\wedge m-1})^T : \Lambda^{m-1} C \to \Lambda^{m-1} B^*$ in place of $T(\alpha)^{-1}$ by identifying $\Lambda^{m-1} C \simeq C^*$, $\Lambda^{m-1} B^* \simeq B$, see [27, §3.8.4] for details.

As a module, as observed in [29], Strassen's degree $m+1$ A-equations are

$$S_{m-1,1,1} A^* \otimes S_{2,1^{m-1}} B^* \otimes S_{2,1^{m-1}} C^*. \tag{2}$$

2.2. **The flag condition.** Much of this paper will use the fact that $T \in A \otimes B \otimes C$ may be recovered up to strict isomorphism from the linear space $T(A^*) \subset B \otimes C$, and if $\dim B = \dim C$ and there exists $\alpha \in A^*$ with $T(\alpha)$ invertible, T may be recovered up to isomorphism from the space $T(A^*)T(\alpha)^{-1} \subset \mathrm{End}(B)$. In this regard, we recall:

Proposition 2.1. [27, Cor. 2.2] *There exist r rank one elements of $B \otimes C$ such that $T(A^*)$ is contained in their span if and only if $\mathbf{R}(T) \leq r$. Similarly, $\underline{\mathbf{R}}(T) \leq r$ if and only if there exists a curve E_t in the Grassmannian $G(r, B \otimes C)$, where for $t \neq 0$, E_t is spanned by r rank one elements and $T(A^*) \subset E_0$ (which is defined by the compactness of the Grassmannian).*

The following two results appeared in [13, Ex. 15.14] and [33, Cor. 18]:

Corollary 2.2. *Let $\mathbf{a} = m$ and let $T \in A \otimes B \otimes C$ be A-concise. Then $\underline{\mathbf{R}}(T) = m$ implies that $T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) \neq \emptyset$.*

Proof. If $E_t \in G(r, B \otimes C)$ is spanned by rank one elements for all $t \neq 0$, then when $t = 0$, it must contain at least one rank one element. But since $\mathbf{a} = m$, $T(A^*) = E_0$. □

Corollary 2.3. *Let $T \in A \otimes B \otimes C$ with $\mathbf{a} = m$ be A-concise. If $\underline{\mathbf{R}}(T) = m$, then there exists a complete flag $A_1 \subset \cdots A_{m-1} \subset A_m = A^*$, with $\dim A_j = j$, such that $\mathbb{P}T(A_j) \subset \sigma_j(Seg(\mathbb{P}B \times \mathbb{P}C))$.*

Proof. Write $T(A^*) = \lim_{t \to 0} \mathrm{span}\{X_1(t), \cdots, X_m(t)\}$ where $X_j(t) \in B \otimes C$ have rank one. Then take $\mathbb{P}A_k = \mathbb{P} \lim_{t \to 0} \mathrm{span}\{X_1(t), \cdots, X_k(t)\} \subset \mathbb{P}T(A^*)$. Since $\mathbb{P}\{X_1(t), \cdots, X_k(t)\} \subset \sigma_k(Seg(\mathbb{P}B \times \mathbb{P}C))$ the same must be true in the limit, and each limit must have the correct dimension because $\dim \lim_{t \to 0} \mathrm{span}\{X_1(t), \cdots, X_m(t)\} = m$. □

Call the implication of Corollary 2.3 the *flag condition*.

There are infinitesimal and scheme-theoretic analogs of Corollary 2.3, but we were unable to state them in general in a useful manner. Here is a special case that indicates the general case. For another example, see §7.4. For a variety $X \subset \mathbb{P}V$, and a smooth point $x \in X$, $\hat{T}_x X \subset V$ denotes its affine tangent space.

Proposition 2.4. *Let $T \in A \otimes B \otimes C$ with $\mathbf{a} = m$ be A-concise. If $\underline{\mathbf{R}}(T) = m$ and $T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) = [X_0]$ is a single point, then $\mathbb{P}(T(A^*) \cap \hat{T}_{[X_0]} Seg(\mathbb{P}B \times \mathbb{P}C))$ must contain a line.*

Call the implication of Proposition 2.4 the *infinitesimal flag condition*.

Proof. Say $T(A^*)$ were the limit of $\mathrm{span}\{X_1(t), \cdots, X_m(t)\}$ with each $X_j(t)$ of rank one. Then since $\mathbb{P}T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) = [X_0]$, we must have each $X_j(t)$ limiting to X_0. But then $\lim_{t \to 0} \mathrm{span}\{X_1(t), X_2(t)\}$, which must be two-dimensional, must be contained in $\hat{T}_{[X_0]} Seg(\mathbb{P}B \times \mathbb{P}C)$ and $T(A^*)$. □

Remark 2.5. Because these conditions deal with intersections, they are difficult to write down as polynomials. We will use them for tensors with simple expressions where they can be checked.

2.3. **Review and clarification of results in** [29]**.** Throughout this subsection we assume $\mathbf{b} = \mathbf{c} = m$. To a 1_A-generic tensor $T \in A \otimes B \otimes C$, fixing $\alpha_0 \in A^*$ as in Definition 1.1, associate a subspace of endomorphisms of B:

$$\mathcal{E}_{\alpha_0}(T) := \{T(\alpha)T(\alpha_0)^{-1} \mid \alpha \in A^*\} \subset \mathrm{End}(B).$$

Note that T may be recovered up to isomorphism from $\mathcal{E}_{\alpha_0}(T)$.

Lemma 2.6. *Let $T \in A \otimes B \otimes C$ be 1_A-generic and assume* $\mathrm{rank}(T(\alpha_0)) = m$.

(1) *If $\underline{\mathbf{R}}(T) = m$ then $\mathcal{E}_{\alpha_0}(T)$ is commutative.*
(2) *If $\mathcal{E}_{\alpha_0}(T)$ is commutative then $\mathcal{E}_{\alpha_0'}(T)$ is commutative for any $\alpha_0' \in A^*$ such that* $\mathrm{rank}(T(\alpha_0')) = m$.

Proof. The first assertion is just a restatement of Strassen's equations. For the second, say $\mathcal{E}_{\alpha_0}(T)$ is commutative, so

$$T(\alpha_1)T(\alpha_0)^{-1}T(\alpha_2) = T(\alpha_2)T(\alpha_0)^{-1}T(\alpha_1) \tag{3}$$

for all $\alpha_1, \alpha_2 \in A^*$. We need to show that

$$T(\alpha_1)T(\alpha_0')^{-1}T(\alpha_2) = T(\alpha_2)T(\alpha_0')^{-1}T(\alpha_1) \tag{4}$$

for all $\alpha_1, \alpha_2 \in A^*$. Since $\mathcal{E}_{\alpha_0}(T)$ is commutative, we have $T(\alpha_0')T(\alpha_0)^{-1}T(\alpha_2) = T(\alpha_2)T(\alpha_0)^{-1}T(\alpha_0')$ and $T(\alpha_0')T(\alpha_0)^{-1}T(\alpha_1) = T(\alpha_1)T(\alpha_0)^{-1}T(\alpha_0')$, i.e., assuming $T(\alpha_1), T(\alpha_2)$ are invertible,

$$T(\alpha_0)^{-1} = T(\alpha_0')^{-1}T(\alpha_j)T(\alpha_0)^{-1}T(\alpha_0')T(\alpha_j)^{-1} \quad \text{for} \quad j = 1, 2.$$

Substituting the $j = 2$ case to the left hand side of (3) and the $j = 1$ case to the right hand side yields (4). The cases where $T(\alpha_j)$ are not invertible follow by taking limits, as the α with $T(\alpha)$ invertible form a Zariski open subset of $T(A^*)$. □

If $U \subset B^* \otimes B$ is commutative, then we may consider it as an abelian Lie-subalgebra of $\mathfrak{gl}(B)$. Define

$$\begin{aligned} Abel_A : &= \mathbb{P}\{T \in A \otimes B \otimes C \mid T \text{ is } A-\text{concise}, \\ &\quad \exists \alpha_0 \in A^* \text{ with } \mathrm{rank}(T(\alpha_0)) = m, \text{ and } \mathcal{E}_{\alpha_0}(T) \subset \mathfrak{gl}(B) \text{ is an abelian Lie algebra}\} \\ &= \mathbb{P}\{T \in A \otimes B \otimes C \mid T \text{ is } A-\text{concise}, \ 1_A-\text{generic and} \\ &\quad \forall \alpha \in A^* \text{ with } \mathrm{rank}(T(\alpha)) = m, \mathcal{E}_{\alpha}(T) \subset \mathfrak{gl}(B) \text{ is an abelian Lie algebra}\} \end{aligned}$$

The second equality follows from Lemma 2.6(2).

Definition 2.7. We say $T \in A \otimes B \otimes C$ is an A-*abelian tensor* if $T \in Abel_A$.

$Abel_A$ is a Zariski closed subset of the set of concise 1_A-generic tensors, namely the zero set of Strassen's equations. Its closure in $A{\otimes}B{\otimes}C$ is a component of the zero set of Strassen's equations.

Define

$$\begin{aligned} Diag_A^0 := \mathbb{P}\{T \in A{\otimes}B{\otimes}C \mid T \text{ is } A-\text{concise}, \\ \exists \alpha_0 \in A^* \text{ with } \operatorname{rank}(T(\alpha_0)) = m, \text{ and } \mathcal{E}_{\alpha_0}(T) \subset \mathfrak{gl}(B) \text{ is diagonalizable}\}, \\ = \mathbb{P}\{T \in A{\otimes}B{\otimes}C \mid T \text{ is } A-\text{concise}, \ 1_A-\text{generic and} \\ \forall \alpha \in A^* \text{ with } \operatorname{rank}(T(\alpha)) = m, \ \mathcal{E}_\alpha(T) \subset \mathfrak{gl}(B) \text{ is diagonalizable}\}. \end{aligned}$$

Let $Diag_A$ be the Zariski closure of $Diag_A^0$ and let $Diag_A^g$ be the intersection of $Diag_A$ with the set of concise 1_A-generic tensors.

Proposition 2.8. [29] *Let $A, B, C = \mathbb{C}^m$, let $T \in A{\otimes}B{\otimes}C$ be concise and 1_A-generic. Then the following are equivalent:*

(1) $\underline{\mathbf{R}}(T) = m$,
(2) $T \in Diag_A^g$.

Moreover, an abelian m-dimensional subspace of $\mathfrak{gl}(B)$ is in the closure of the diagonalizable subspaces if and only if it arises as $\mathcal{E}_\alpha(T)$ for some concise, border rank m tensor $T \in A{\otimes}B{\otimes}C$.

Proof. Since the proof in the literature is not explicit and we use it frequently, we show that if $T \in A{\otimes}B{\otimes}C$ is concise, then $\mathcal{E}_\alpha(T)$ belongs to the limit of diagonalizable subalgebras. We know there exists a curve of m-tuples of rank one tensors $(T_i^t)_{i=1}^m$ such that in the Grassmannian

$$\lim_{t\to 0} \operatorname{span}\{T_i^t(A^*)\} \to T(A^*).$$

In particular, there exist $X_t \in \operatorname{span}\{T_i^t(A^*)\}$, such that $X_t \to T(\alpha)$ and we may assume that X_t are invertible. Then, $\operatorname{span}\{T_i^t(A^*)\}X_t^{-1}$ is a curve of diagonalizable algebras converging to $\mathcal{E}_\alpha(T)$. ☐

2.4. **The** End**-closed condition.** Throughout this subsection we assume $\mathbf{b} = \mathbf{c} = m$. Define

$$End-Abel_A := \mathbb{P}\{T \in Abel_A \mid \exists \alpha \in A^* \text{ with } \operatorname{rank}(T(\alpha)) = m, \text{ and } \mathcal{E}_\alpha(T) \text{ is closed under composition}\}.$$

Remark 2.9. There was ambiguity in the definition of $Comm_{\mathbf{a},\mathbf{b}}$ in [29] that is clarified by the above notions which replace it.

The following Proposition essentially dates back to Gerstenhaber [21]. It is utilized in [33, §5] to obtain explicit abelian subspaces that are not in $Red(m)$, see §5.1.

Proposition 2.10. *If $T \in A{\otimes}B{\otimes}C$, with $\mathbf{b} = \mathbf{c} = m$ is 1_A-generic, $\underline{\mathbf{R}}(T) = m$ and* $\operatorname{rank}(T(\alpha_0)) = m$*, then $\mathcal{E}_{\alpha_0}(T)$ is closed under composition.*

Proof. Each diagonalizable Lie algebra is closed under composition. The property of being closed under composition is a Zariski closed property. ☐

Proposition 2.11. *Let $T \in End-Abel_A$ and* $\operatorname{rank}(T(\alpha_0)) = m$*. Then $\mathcal{E}_{\alpha_0}(T) = \mathcal{E}_{\alpha_0'}(T)$ for any $\alpha_0' \in A^*$ such that* $\operatorname{rank}(T(\alpha_0')) = m$*. In particular,*

$$End{-}Abel_A = \mathbb{P}\{T \in Abel_A \mid \forall \alpha \in A^* \text{ with } \operatorname{rank}(T(\alpha)) = m, \mathcal{E}_\alpha(T) \text{ is closed under composition}\}.$$

Proof. By the End-closed condition applied to α_0', for any $\alpha \in A^*$ there exists $\alpha' \in A^*$ such that

$$T(\alpha)T(\alpha_0')^{-1}T(\alpha_0)T(\alpha_0')^{-1} = T(\alpha')T(\alpha_0')^{-1}$$

i.e.,

$$T(\alpha)T(\alpha_0')^{-1} = T(\alpha')T(\alpha_0')^{-1}T(\alpha_0')T(\alpha_0)^{-1} = T(\alpha')T(\alpha_0)^{-1}$$

so $\mathcal{E}_{\alpha_0'}(T) \subseteq \mathcal{E}_{\alpha_0}(T)$. As both spaces are of the same dimension, equality must hold. □

Note the inclusions

$$Diag_A^g \subseteq End - Abel_A \subseteq Abel_A.$$

These spaces all coincide when $m \leq 4$, $Diag_A^g = End - Abel_A$ for $m = 5$, $End - Abel_A \subsetneq Abel_A$ when $m \geq 5$ (see §5.1), and are all different when $m \geq 7$ (see §5.2).

Proposition 2.12. *The subvariety $End - Abel_A$ in the set of 1_A-generic tensors has equations that as a $GL(A) \times GL(B) \times GL(C)$-module include*

$$S_{m,3,1^{m-2}}A^* \otimes \Big(\bigoplus_{\substack{|\pi|=m+1 \\ p_1, \ell(\pi) \leq m}} S_{\pi+(1^m)}B^* \otimes S_{\pi'+(1^m)}C^* \Big).$$

These equations are of degree $2m+1$.

Proof. For $\alpha_1, \cdots, \alpha_m$ a basis of A^*, if $T \in End - Abel_A$, then for all $\alpha, \alpha' \in A^*$,

$$T(\alpha)(T(\alpha_1)^{\wedge m-1})^T T(\alpha') \subset \operatorname{span}\{T(\alpha_1), \cdots, T(\alpha_m)\}.$$

In other words, the following vector in $\Lambda^{m+1}(B \otimes C)$ must be zero:

$$T(\alpha)(T(\alpha_1)^{\wedge m-1})^T T(\alpha') \wedge T(\alpha_1) \wedge \cdots \wedge T(\alpha_m).$$

The entries of this vector are polynomials of degree $2m+1$ in the coefficients of T, as the entries of $(T(\alpha_1)^{\wedge m-1})^T$ are of degree $m-1$ in the coefficients of T and all the other matrices have entries that are linear in the coefficients of T. Among the quantities that must be zero are the coefficients of $b_1 \otimes c_1 \wedge \cdots \wedge b_1 \otimes c_m \wedge b_2 \otimes c_1$, and more generally the coefficients of $b_1 \otimes c_1 \wedge \cdots \wedge b_1 \otimes c_{q_{p_1}} \wedge \cdots \wedge b_{p_1} \otimes c_1 \wedge \cdots \wedge b_{p_1} \otimes c_{q_{p_1}}$ where $\pi = (p_1, \cdots, p_{q_1})$ is a partition of $m+1$ with first part at most m, $q_1 \leq m$, and $\pi' = (q_1, \cdots, q_{p_1})$. Now take $\alpha, \alpha' = \alpha_2$, the corresponding coefficients have the stated weight and all are highest weight vectors. □

3. The Alexeev-Forbes-Tsimerman method for bounding tensor rank

Because the set of tensors of rank at most r is not closed, there are few techniques for proving lower bounds on rank that are not just lower bounds for border rank. What follows is the only general technique we are aware of. (However for very special tensors like matrix multiplication, additional methods are available, see [28].) The method below, generally called the *substitution method* was introduced in [36] and used in [47, 42] among other places. We follow the novel application of it from [1]. Fix a basis $a_1, \cdots, a_{\mathbf{a}}$ of A. Write $T = \sum_{i=1}^{\mathbf{a}} a_i \otimes M_i$, where $M_i \in B \otimes C$.

Proposition 3.1. [1, Appendix B], [6, Chapter 6] *Let $\mathbf{R}(T) = r$ and $M_1 \neq 0$. Then there exist constants $\lambda_2, \ldots, \lambda_{\mathbf{a}}$, such that the tensor*

$$\tilde{T} := \sum_{j=2}^{\mathbf{a}} a_j \otimes (M_j - \lambda_j M_1) \in a_1{}^{\perp} \otimes B \otimes C,$$

has rank at most $r-1$. Moreover, if $\operatorname{rank}(M_1) = 1$ *then for any choice of $(\lambda_2, \cdots, \lambda_{\mathbf{a}})$ we have $\mathbf{R}(\tilde{T}) \geq r-1$.*

The statement of Proposition 3.1 is slightly different from the original statement in [1], so we give a modified proof:

Proof. By Proposition 2.1 there exist rank one elements $X_1, \dots, X_r \in B \otimes C$ and scalars d_j^u such that:

$$M_j = \sum_{u=1}^{r} d_j^u X_u.$$

Since $M_1 \neq 0$ we may assume $d_1^1 \neq 0$ and define $\lambda_j = \frac{d_j^1}{d_1^1}$. Then the subspace $\tilde{T}({a_1}^{\perp})$ is spanned by $X_2, \dots, X_r$ so Proposition 2.1 implies $\mathbf{R}(\tilde{T}) \leq r - 1$. The last assertion holds because if $\operatorname{rank}(M_1) = 1$ then we may assume $X_1 = M_1$. □

Proposition 3.1 is usually implemented by consecutively applying the following steps, which we will refer to as the substitution method:

(1) Distinguish A, take a basis $\{a_j\}$ of it and take bases$\{\beta_s\}$, $\{\gamma_t\}$ of B^*, C^* and represent T as a matrix M with entries that are linear combinations of the basis vectors a_i: $M_{s,t} = T(\beta_s \otimes \gamma_t)$.
(2) Choose a subset of $\mathbf{b}'$ columns of M and $\mathbf{c}'$ rows of M.
(3) Inductively, for elements of the chosen columns (resp. rows) remove the nonzero u-th column (resp. row) and add to all other columns (resp. rows) the u-th column (resp. row) times an arbitrary coefficient λ, regarding the a_j as formal variables. This step is just to ensure that each time only nonzero columns or rows are removed.
(4) Set all a_j that appeared in any of the selected rows or columns to zero, obtaining a matrix M'. Notice, that M' does not depend on the choice of λ.
(5) The rank of T is at least $\mathbf{b}'$ plus $\mathbf{c}'$ plus the rank of the tensor corresponding to M'.

The above steps can be iterated, interchanging the roles of A, B and C.

Example 3.2. [1] Let

$$T(A^*) = \begin{pmatrix} x_1 & & & & & & & \\ & x_1 & & & & & & \\ & & x_1 & & & & & \\ & & & x_1 & & & & \\ x_2 & & & & x_1 & & & \\ & x_2 & & & & x_1 & & \\ x_3 & & x_2 & & & & x_1 & \\ x_4 & x_3 & & x_2 & & & & x_1 \end{pmatrix},$$

where here, and in what follows, blank entries are zero. Then $\mathbf{R}(T) \geq 15$. Indeed, in the first iteration of the method presented above, choose the first four rows and last four columns. One obtains a 4×4 matrix M' and the associated tensor T', so $\mathbf{R}(T) \geq 8 + \mathbf{R}(T')$. Iterating the method twice yields $\mathbf{R}(T) \geq 8 + 4 + 2 + 1 = 15$.

On the other hand $\underline{\mathbf{R}}(T) = 8$, e.g., because $T(A^*)$, after a choice of α^1, is a specialization of a space that is abelian and contains a regular nilpotent element and we conclude by Corollary 6.2 below.

To see that $\mathbf{R}(T) = 15$, one can construct an explicit expression or appeal to Proposition 6.3 because $T(A^*)$ is a degeneration of the centralizer of a regular nilpotent element.

This generalizes to $T \in \mathbb{C}^k \otimes \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$ of rank $2 * 2^k - 1$ and border rank 2^k.

Example 3.3. [1] Let $T = a_1{\otimes}(b_1{\otimes}c_1{+}\cdots{+}b_8{\otimes}c_8){+}a_2{\otimes}(b_1{\otimes}c_5{+}b_2{\otimes}c_6{+}b_3{\otimes}c_7{+}b_4{\otimes}c_8){+}a_3{\otimes}(b_1{\otimes}c_7{+}b_2{\otimes}c_8) + a_4{\otimes}b_1{\otimes}c_8 + a_5{\otimes}b_8{\otimes}c_1 + a_6{\otimes}b_8{\otimes}c_2 + a_7{\otimes}b_8{\otimes}c_3 + a_8{\otimes}b_8{\otimes}c_4$, so

$$T(A^*) = \begin{pmatrix} x_1 & & & & & & & x_5 \\ & x_1 & & & & & & x_6 \\ & & x_1 & & & & & x_7 \\ & & & x_1 & & & & x_8 \\ x_2 & & & & x_1 & & & \\ & x_2 & & & & x_1 & & \\ x_3 & & x_2 & & & & x_1 & \\ x_4 & x_3 & & x_2 & & & & x_1 \end{pmatrix}.$$

Then $\mathbf{R}(T) \geq 18$. Here we start by contracting x_5, x_6, x_7, x_8. We obtain a tensor $\tilde{T}$ represented by the matrix

$$\begin{pmatrix} x_1 & & & & & & & \\ & x_1 & & & & & & \\ & & x_1 & & & & & \\ & & & x_1 & & & & \\ x_2 & & & & x_1 & & & \\ & x_2 & & & & x_1 & & \\ x_3 & & x_2 & & & & x_1 & \\ x_4 & x_3 & & x_2 & & & & \end{pmatrix},$$

and $\mathbf{R}(T) \geq 4 + \mathbf{R}(\tilde{T})$. The substitution method then gives $\mathbf{R}(\tilde{T}) \geq 14$. In fact, $\mathbf{R}(T) = 18$; it is enough to consider 17 matrices with just one nonzero entry corresponding to all nonzero entries of $T(A^*)$, apart from the top left and bottom right corner and one matrix with 1 at each corner and all other entries equal to 0. This generalizes to $T \in \mathbb{C}^{2^k}{\otimes}\mathbb{C}^{2^k}{\otimes}\mathbb{C}^{2^k}$ of rank $3 * 2^k - k - 3$.

For tensors in $\mathbb{C}^m{\otimes}\mathbb{C}^m{\otimes}\mathbb{C}^m$, the limit of the method would be to prove a tensor has rank at least $3(m-1)$, and this can be achieved only by exchanging the roles of A, B, C in the application successively.

4. A Remark on Strassen's additivity conjecture

Strassen's additivity conjecture [43] states that the rank of the sum of two tensors in disjoint spaces equals the sum of the ranks. While this conjecture has been studied from several different perspectives, e.g. [17, 25, 8, 14, 9], very little is known about it, and experts are divided as to whether it should be true or false.

In many cases of low rank the substitution method provides the correct rank. In light of this, the following theorem indicates why providing a counter-example to Strassen's conjecture may be difficult.

Theorem 4.1. *Let $T_1 \in A_1{\otimes}B_1{\otimes}C_1$ and $T_2 \in A_2{\otimes}B_2{\otimes}C_2$ be such that that $\mathbf{R}(T_1)$ can be determined by the substitution method. Then Strassen's additivity conjecture holds for $T_1 \oplus T_2$, i.e, $\mathbf{R}(T_1 \oplus T_2) = \mathbf{R}(T_1) + \mathbf{R}(T_2)$.*

Proof. With each application of the substitution method, T_1 is modified to a tensor of lower rank living in a smaller space and T_2 is unchanged. After all applications, T_1 has been modified to zero and T_2 is still unchanged. □

The rank of any tensor in $\mathbb{C}^2{\otimes}B{\otimes}C$ can be computed using the substitution method as follows: by dimension count, we can always find either $\beta \in B^*$ or $\gamma \in C^*$, such that $T(\beta)$ or $T(\gamma)$ is a rank one matrix. In particular, Theorem 4.1 provides an easy proof of Strassen's additivity conjecture if the dimension of any of A_1, B_1 or C_1 equals 2. This was first shown in [25] by other methods and is further investigated by Buczyński and Postinghel, see [12].

5. Abelian tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with border rank greater than m

5.1. *End* − *Abel*$_A \subsetneq$ *Abel*$_A$ **for** $m \geq 5$**.** The lower bounds for border rank in the following two propositions appeared in [33, Def. 16] in the language of groups. It answers [24, Question A] and provides an example asked for in [24, Remark after Question B] and a concise 1-generic tensor $T \in \mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ with $\underline{\mathbf{R}}(T) = 6$ satisfying Strassen's equations.

Proposition 5.1. *Let* $T_{Leit,5} = a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3 + b_4 \otimes c_4 + b_5 \otimes c_5) + a_2 \otimes (b_1 \otimes c_3 + b_3 \otimes c_5) + a_3 \otimes b_1 \otimes c_4 + a_4 \otimes b_2 \otimes c_4 + a_5 \otimes b_2 \otimes c_5$, *which gives rise to the linear space*

$$T_{Leit,5}(A^*) = \begin{pmatrix} x_1 & & & & \\ 0 & x_1 & & & \\ x_2 & 0 & x_1 & & \\ x_3 & x_4 & 0 & x_1 & \\ 0 & x_5 & x_2 & 0 & x_1 \end{pmatrix}.$$

Then $\mathcal{E}_{\alpha^1}(T_{Leit,5})$ *is an abelian Lie algebra, but not* End*-closed. I.e.,* $T_{Leit,5} \in Abel_A$ *but* $T_{Leit,5} \notin End - Abel_A$. *In particular,* $T_{Leit,5} \notin Diag_A$ *so* $\underline{\mathbf{R}}(T_{Leit,5}) > 5$. *In fact,* $\underline{\mathbf{R}}(T_{Leit,5}) = 6$ *and* $\mathbf{R}(T_{Leit,5}) = 9$.

Proof. The first statements are verifiable by inspection. The fact that the border rank of the tensor is at least 6 follows from Theorem 2.8. The fact that border rank equals 6 follows by considering rank one matrices:

$$X_1 = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 1 & 1 & & & \\ 1 & 1 & & & \end{pmatrix}, X_2 = \begin{pmatrix} & & & & \\ & & & & \\ \epsilon & & \epsilon^2 & & \\ & & & & \\ 1 & & \epsilon & & \end{pmatrix}, X_3 = \begin{pmatrix} \epsilon^2 & & & & \epsilon^4 \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \epsilon^2 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} & & & & \\ & & & & \\ & \epsilon & \epsilon^2 & & \\ & 1 & \epsilon & & \\ & & & & \end{pmatrix}, X_5 = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 1 & & -\epsilon & \epsilon^2 & \\ & & & & \end{pmatrix}, X_6 \begin{pmatrix} & & & & \\ & \epsilon^2 & & & \\ & -\epsilon & & & \\ & & & & \\ & 1 & & & \end{pmatrix}.$$

Then

$$T_{Leit,5} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} a_1 \otimes (-X_1 + X_3 + X_4 + X_5 + X_6) + \frac{1}{\epsilon} a_2 \otimes (X_2 - X_3) + a_3 \otimes X_5 + a_4 \otimes X_4 + a_5 \otimes X_6,$$

which is a sum of six rank one tensors. In terms of tensor products,

$$\begin{aligned} T_{Leit,5} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} [& -a_1 \otimes (b_1 + b_2) \otimes (c_4 + c_5) + \epsilon a_2 \otimes (b_1 + \epsilon b_3) \otimes (c_5 + \epsilon c_3) \\ & + (a_1 - \epsilon a_2) \otimes (b_1 + \epsilon^2 b_5) \otimes (c_5 + \epsilon^2 c_1) + (a_1 + \epsilon^2 a_4) \otimes (b_2 + \epsilon b_3) \otimes (c_4 + \epsilon c_3) \\ & + (a_1 + \epsilon^2 a_3) \otimes (b_1 - \epsilon b_3 + \epsilon^2 b_4) \otimes c_4 + (a_1 + \epsilon^2 a_5) \otimes b_2 \otimes (c_5 - \epsilon c_3 + \epsilon^2 c_2)]. \end{aligned}$$

The substitution method shows that $\mathbf{R}(T_{Leit,5}) \geq 9$. To prove equality, consider the 9 rank 1 matrices:

(1) 3 matrices with just one nonzero entry corresponding to x_3, x_4, x_5,

(2) The six matrices

$$\begin{pmatrix} 1 & & & & \\ & & & & \\ -1 & & & & \\ & & & & \\ -1 & & & & \end{pmatrix}, \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & -1 & & 1 \end{pmatrix}, \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & 1 & & \\ & & & & \end{pmatrix}, \begin{pmatrix} & & & & \\ & 1 & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}, \begin{pmatrix} & & & & \\ & & & & \\ 1 & & 1 & & \\ & & & & \\ 1 & & 1 & & \end{pmatrix}, \begin{pmatrix} & & & & \\ & & & & \\ 1 & & -1 & & \\ & & & & \\ -1 & & 1 & & \end{pmatrix}.$$

$T(A^*)$ is contained in the span of these matrices. □

Note that $T_{Leit,5}$ is neither 1_B nor 1_C-generic. The example easily generalizes to higher m, e.g. for $m = 7$ we could take:

$$T_{Leit,7}(A^*) = \begin{pmatrix} x_1 & & & & & & \\ & x_1 & & & & & \\ & & x_1 & & & & \\ x_2 & & & x_1 & & & \\ x_3 & & & & x_1 & & \\ x_4 & x_7 & & & & x_1 & \\ & x_5 & x_6 & x_2 & & & x_1 \end{pmatrix}.$$

Proposition 5.2. *The following tensor:*

$T'_{Leit,6} = a_1 \otimes (b_1 \otimes c_1 + \dots + b_6 \otimes c_6) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3) + a_3 \otimes b_1 \otimes c_5 + a_4 \otimes b_1 \otimes c_6 + a_5 \otimes b_4 \otimes c_5 + a_6 \otimes b_4 \otimes c_6,$

which gives rise to the abelian subspace:

$$T'_{Leit,6}(A^*) = \begin{pmatrix} x_1 & & & & & \\ x_2 & x_1 & & & & \\ & x_2 & x_1 & & & \\ & & & x_1 & & \\ x_3 & & & x_5 & x_1 & \\ x_4 & & & x_6 & & x_1 \end{pmatrix},$$

is not End*-closed and satisfies* $\underline{\mathbf{R}}(T'_{Leit,6}) = 7$ *and* $\mathbf{R}(T'_{Leit,6}) = 11$.

Proof. The border rank is at least 7 as $T'_{Leit,6}(A^*)$ is not End-closed. The rank is at least 11 by the substitution method.

To prove that rank is indeed 11 we notice that the 3×3 upper square of $T'_{Leit,6}(A^*)$

$$\begin{pmatrix} x_1 & & \\ x_2 & x_1 & \\ & x_2 & x_1 \end{pmatrix}$$

represents a tensor of rank at most 4 by considering:

$$\begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & 1 & \\ 1 & 1 & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & -1 & \\ -1 & 1 & \end{pmatrix}, \begin{pmatrix} 1 & & -1 \\ & & \\ -1 & & 1 \end{pmatrix}.$$

Apart from this square there are 7 nonzero entries, so the rank is at most $7 + 4 = 11$.

To compute the border rank notice that after removing the second row and column we obtain a tensor of border rank 5 by Proposition 6.4 below. On the other hand the entries in the second column and row clearly form a border rank 2 tensor. In other words, the tensor corresponding to

$$\begin{pmatrix} x_1 & & & & & & \\ x_2 & x_7 & & & & & \\ & x_2 & x_1 & & & & \\ & & & x_1 & & & \\ x_3 & & & x_5 & x_1 & & \\ x_4 & & & x_6 & & x_1 \end{pmatrix}.$$

has border rank 7 and $T'_{Leit,6}$ is a specialization if it. ☐

5.2. $Diag^g_A \subsetneq End-Abel_A$ **for** $m \geq 7$**.**

Proposition 5.3. *The tensor corresponding to*

$$T_{end,7}(A^*) = \begin{pmatrix} x_1 & & & & & & & \\ & x_1 & & & & & & \\ & & x_1 & & & & & \\ & & & x_1 & & & & \\ & x_2+x_7 & x_3 & x_4 & x_1 & & & \\ x_2 & x_3 & x_5 & x_6 & & x_1 & \\ x_4 & x_5 & x_6 & x_7 & & & x_1 \end{pmatrix}$$

is End*-closed, but has border rank at least* 8*.*

Proof. The fact that it is End-closed follows by inspection. The tensor has border rank at least 8 by Corollary 2.2 as $T_{end,7}(A^*)$ does not intersect the Segre. Indeed, if it intersected Segre we would have $x_1 = x_4 = 0$, and $(x_2+x_7)x_2 = 0$. If $x_2 = 0$, then $x_7^2 = (x_2+x_7)x_7 = 0$, which implies $x_3 = x_5 = x_6 = 0$ and gives a contradiction. If $x_2 + x_7 = 0$ we obtain $x_3 = x_5 = x_6 = 0$. This implies $x_7 = 0$ and leads to a contradiction. ☐

The following proposition yields families of *End*-closed tensors of large rank for large m and border rank greater than m as soon as $m \geq 8$:

Proposition 5.4. *Let* $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ *be such that* $T = a_1 \otimes (b_1 \otimes c_1 + \cdots b_m \otimes c_m) + T'$ *where* $T' \in A' \otimes B' \otimes C' := \mathrm{span}\{a_2, \cdots, a_m\} \otimes \mathrm{span}\{b_1, \cdots, b_{\lfloor \frac{m}{2} \rfloor}\} \otimes \mathrm{span}\{c_{\lceil \frac{m}{2} \rceil}, \cdots, c_m\}$*. Then* $\mathbf{R}(T) = \mathbf{R}(T') + m$*.*

Proof. We have

$$T(A^*) \subset \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_1 & & & \\ * & \cdots & * & x_1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ * & \cdots & * & & & x_1 \end{pmatrix}.$$

The lower bound on rank is obtained by substitution method and the upper bound follows as rank is sub-additive. ☐

Corollary 5.5. *A general element of* $End-Abel_A$ *has border rank at least* $\frac{m^2}{8}$*, which is approximately* $\frac{3}{8}$*-of the maximal border rank of a general tensor in* $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$*.*

Proof. $\underline{\mathbf{R}}(T) \geq \underline{\mathbf{R}}(T') \geq \frac{m^2}{8}$ for general $T' \in \mathbb{C}^{m-1} \otimes \mathbb{C}^{\lfloor \frac{m}{2} \rfloor} \otimes \mathbb{C}^{\lceil \frac{m}{2} \rceil}$. ☐

5.3. **A generic A-abelian,** End**-closed tensor satisfying** $\mathbb{P}T(A^*)\cap Seg(\mathbb{P}B\times\mathbb{P}C)\neq\emptyset$ **has high border rank.**

Proposition 5.6. *Let $m-1=k+\ell$ with $3(k+\ell)<k\ell+5$. Then an abelian tensor $T\in\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ of the form $T=M_{\langle 1\rangle}\oplus T'$ with $M_{\langle 1\rangle}\in\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1$ and $T'\in\mathbb{C}^{m-1}\otimes\mathbb{C}^{m-1}\otimes\mathbb{C}^{m-1}=A'\otimes B'\otimes C'$ of the form $a_1\otimes(b_1\otimes c_1+\cdots+b_{m-1}\otimes c_{m-1})+T''$ with $T''\in\mathbb{C}^{m-2}\otimes\mathbb{C}^k\otimes\mathbb{C}^\ell$ general, where $\mathbb{C}^k=$ span$\{b_1,\cdots,b_k\}$ and $\mathbb{C}^\ell=$ span$\{c_k+1,\cdots,c_{m-1}\}$, has $\underline{\mathbf{R}}(T)>m$ despite being 1_A-generic, abelian,* End*-closed, and such that $\mathbb{P}T(A^*)\cap Seg(\mathbb{P}B\times\mathbb{P}C)\neq\emptyset$.*

Proof. For general T'' the space $T''((\mathbb{C}^{m-2})^*)\in\mathbb{C}^k\otimes\mathbb{C}^\ell$ will not intersect $\sigma_2(Seg(\mathbb{P}^{k-1}\times\mathbb{P}^{\ell-1}))$ if $\dim\mathbb{P}T''(A^*)+\dim\sigma_2(Seg(\mathbb{P}^{k-1}\times\mathbb{P}^{\ell-1}))<\dim\mathbb{P}(\mathbb{C}^k\otimes\mathbb{C}^\ell)$, i.e., if $[2(k+\ell-2)-1]+[k+\ell-1]<k\ell-1$, i.e., if $3(k+\ell)<k\ell+5$, so T violates the flag condition. □

5.4. **A tensor in** $End-Abel_8$ **of border rank** >8 **via Corollary 2.2.** Consider (from [33, Prop. 19])

$$\begin{aligned}T''_{Leit,8}=&a_1\otimes(b_1\otimes c_1+\cdots+b_8\otimes c_8)+a_2\otimes(b_4\otimes c_5+b_3\otimes c_6)+a_3\otimes(b_3\otimes c_5+b_2\otimes c_6)+a_4\otimes(b_2\otimes c_5+b_1\otimes c_6)\\&+a_5\otimes(b_4\otimes c_7+b_3\otimes c_8)+a_6\otimes(b_3\otimes c_7+b_2\otimes c_8)+a_7\otimes(b_2\otimes c_7+b_1\otimes c_8)+a_8\otimes(b_1\otimes c_7+b_4\otimes c_6)\end{aligned}$$

so

$$T''_{Leit,8}(A^*)=\begin{pmatrix}x_1&&&&&&&\\&x_1&&&&&&\\&&x_1&&&&&\\&&&x_1&&&&\\&x_4&x_3&x_2&x_1&&&\\x_4&x_3&x_2&x_8&&x_1&&\\x_8&x_7&x_6&x_5&&&x_1&\\x_7&x_6&x_5&&&&&x_1\end{pmatrix}.$$

Since $\mathbb{P}T(A^*)\cap Seg(\mathbb{P}B\times\mathbb{P}C)=\emptyset$, $\underline{\mathbf{R}}(T''_{Leit,8})>8$. In Leitner's language the bound arises because the corresponding group does not contain a one parameter subgroup of rank one.

5.5. **A-abelian tensors that intersect the Segre and fail to satisfy flag condition.** The following example answers a question about sufficient conditions to be a limit of diagonalizable groups in [33, p. 10].

Example 5.7. Consider $T_{s9}:=M_{\langle 1\rangle}\oplus T''_{Leit,8}\in\mathbb{C}^9\otimes\mathbb{C}^9\otimes\mathbb{C}^9=A\otimes B\otimes C$. To keep indices consistent with above, write $M_{\langle 1\rangle}=a_0\otimes b_0\otimes c_0$, so they range from 0 to 8. Note that $T_{s9}\cap Seg(\mathbb{P}^8\times\mathbb{P}^8\times\mathbb{P}^8)\neq\emptyset$, and T_{s9} is 1_A-generic and abelian, however $\underline{\mathbf{R}}(T_{s9})>9$. To see this, note that the flag condition fails because there is no $\mathbb{P}^1$ contained in $\sigma_2(Seg(\mathbb{P}^8\times\mathbb{P}^8))$.

5.6. **An End-closed tensor satisfying the flag condition but not the infinitesimal flag condition.** Consider

$$T_{flagok}(A^*):=\begin{pmatrix}x_1&&&&&&&&\\x_0&x_1&&&&&&&\\&&x_1&&&&&&\\&&&x_1&&&&&\\&&&&x_1&&&&\\&&x_4&x_3&x_2&x_1&&&\\x_4&&x_3&x_2&x_8&&x_1&&\\x_8&&x_7&x_6&x_5&&&x_1&\\x_7&&x_6&x_5&&&&&x_1\end{pmatrix}$$

Here $\mathbb{P}T_{flagok}(A^*) \cap Seg(\mathbb{P}^8 \times \mathbb{P}^8) = [X_0]$, where X_0 is the matrix with 1 in the $(2,1)$ entry and zero elsewhere, and

$$\hat{T}_{[X_0]}Seg(\mathbb{P}^8 \times \mathbb{P}^8) = \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ * & 0 & \cdots & 0 \end{pmatrix}$$

which only intersects $T_{flagok}(A^*)$ in $[X_0]$. Thus by Proposition 2.4, $\underline{\mathbf{R}}(T_{flagok}) > 9$.

The flag condition is satisfied: consider respectively spaces spanned by $x_0, x_4, x_3, x_2, x_8, x_7, x_6, x_5$. It straightforward to check that T_{flagok} is End-closed.

5.7. **1-generic abelian tensors.** In general, the spaces $T(A^*), T(B^*), T(C^*)$ can be very different, e.g. if T is 1_A-generic and not 1_B-generic, or if T is not abelian. The following proposition is a variant of remarks in [27, §7.7.2] and [19]:

Proposition 5.8. *Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be 1_A and 1_B generic and satisfy the A-Strassen's equations. Then T is isomorphic to a tensor in $S^2\mathbb{C}^m \otimes \mathbb{C}^m$.*

In particular:

1. *After making choices of general $\alpha \in A^*$ and $\beta \in B^*$, $T(A^*)$ and $T(B^*)$ are GL_m-isomorphic subspaces of* $\mathrm{End}(\mathbb{C}^m)$.
2. *If T is 1-generic, then T is isomorphic to a tensor in $S^3\mathbb{C}^m$.*

Proof. Let $\{a_i\}, \{b_j\}, \{c_j\}$ respectively be bases of A, B, C. Write $T = \sum t^{ijk} a_i \otimes b_j \otimes c_k$. Possibly after a change of basis we may assume $t^{1jk} = \delta_{jk}$ and $t^{i1k} = \delta_{ik}$. Take $\{\alpha^i\}$ the dual basis to $\{a_j\}$ and identify $T(A^*) \subset \mathrm{End}(\mathbb{C}^m)$ via α^1. Strassen's A-equations then say

$$0 = [T(\alpha^{i_1}), T(\alpha^{i_2})]_{(j,k)} = \sum_l t^{i_1 j l} t^{i_2 l k} - t^{i_2 j l} t^{i_1 l k} \ \forall i_1, i_2, j, k.$$

Consider when $j = 1$:

$$0 = \sum_l t^{i_1 1 l} t^{i_2 l k} - t^{i_2 1 l} t^{i_1 l k} = t^{i_2 i_1 k} - t^{i_1 i_2 k} \ \forall i_1, i_2, k,$$

because $t^{i_1 1 l} = \delta_{i_1, l}$ and $t^{i_2 1 l} = \delta_{i_2, l}$. But this says $T \in S^2\mathbb{C}^m \otimes \mathbb{C}^m$.

For the last assertion, say $L_B : B \to A$ is such that $Id_A \otimes L_B \otimes Id_C(T) \in S^2 A \otimes C$ and $L_C : C \to A$ is such that $Id_A \otimes Id_B \otimes L_C \in S^2 A \otimes B$. Then $Id_A \otimes L_B \otimes L_C(T)$ is in $A^{\otimes 3}$, symmetric in the first and second factors as well as the first and third. But $\mathfrak{S}_3$ is generated by two transpositions, so $Id_A \otimes L_B \otimes L_C(T) \in S^3 A$. □

Thus the A, B, C-Strassen's equations, despite being very different modules, when restricted to 1-generic tensors, all have the same zero sets. Strassen's equations in the case of partially symmetric tensors were essentially known to Emil Toeplitz [45], and in the symmetric case to Aronhold [3].

Proposition 5.9. *There exist 1-generic abelian tensors $T \in \mathbb{C}^{4k} \otimes \mathbb{C}^{4k} \otimes \mathbb{C}^{4k}$ that have border rank at least $\frac{k(k+1)}{6}$. In particular, there exist tensors in $\mathbb{C}^{4k} \otimes \mathbb{C}^{4k} \otimes \mathbb{C}^{4k}$ satisfying the Strassen-Aronhold equations with $\underline{\mathbf{R}}(T) \geq \frac{k(k+1)}{6}$.*

Proof. We exhibit a family of $4k$-dimensional subspaces of $S^2\mathbb{C}^{4k}$ whose associated tensor can degenerate to a general tensor $T' \in S^2\mathbb{C}^k \otimes \mathbb{C}^{k-1}$. Since the border rank of a general tensor in

$S^2\mathbb{C}^k \otimes \mathbb{C}^{k-1} \subset \mathbb{C}^k \otimes \mathbb{C}^k \otimes \mathbb{C}^{k-1}$ must satisfy:

$$\dim \sigma_{\underline{\mathbf{R}}(T')}(Seg(\mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \times \mathbb{P}^{k-2})) > \dim \mathbb{P}(S^2\mathbb{C}^k \otimes \mathbb{C}^{k-1})) = \frac{(k-1)k(k+1)}{2} - 1,$$

and $\dim \sigma_r(Seg(\mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \times \mathbb{P}^{k-2})) = r(3k-4) + r - 1$ (by [34]), we obtain:

$$\underline{\mathbf{R}}(T) \geq \underline{\mathbf{R}}(T') \geq \frac{k(k+1)}{6}.$$

Represent T as a $4k \times 4k$ symmetric matrix M with entries that are linear functions of $4k$ variables. The variables will play four different roles, so we name them accordingly:

(1) variable z,
(2) variables $y_1, \dots, y_{k-1}$,
(3) variables $x_1, \dots, x_k$,
(4) variables $z_1, \dots, z_{2k}$.

They appear in the given order (starting from the top) in the first column of M, ensuring 1_B-genericity. This also defines the last row of M, ensuring 1_C-genericity. (Here we order the bases of B and C in opposite order to have the matrix symmetric with respect to an anti-diagonal reflection.) The matrix M will be lower triangular, with the variable z on the diagonal (ensuring 1_A-genericity), and z appears only on the diagonal. The variables $z_1, \dots, z_{2k}$ appear only in the last row and first column as defined previously.

Write a general tensor $T' \in S^2\mathbb{C}^k \otimes \mathbb{C}^{k-1}$ as $T' = \sum a^l_{ij} e_i \otimes e_j \otimes f_l$, where $a^l_{ij} = a^l_{ji}$. We use T' to define the entries in rows $3k+1, \dots, 4k-1$ and columns $k+1, \dots, 2k$: let $\sum_{i=1}^k a^u_{is} x_i$ be the entry in the $(3k+u, k+s)$-position. As M is symmetric this defines also the entries in rows $2k+1, \dots, 3k$ and columns $2, \dots, k$.

Apart from entries defined so far, the only remaining nonzero entries belong to a $k \times k$ submatrix of rows from $2k+1, \dots, 3k$ and columns $k+1, \dots, 2k$: the linear form in the y's $\sum_{i=1}^{k-1} a^{k-i}_{(k+1-u),s} y_i$ is the $(2k+u, k+s)$-entry.

For example, when $k = 3$,

$$T(A^*) = \begin{pmatrix}
z &&&&&&&&&&& \\
y_1 & z &&&&&&&&&& \\
y_2 && z &&&&&&&&& \\
x_1 &&& z &&&&&&&& \\
x_2 &&&& z &&&&&&& \\
x_3 &&&&& z &&&&&& \\
z_1 & \sum_i a^2_{3i}x_i & \sum_i a^1_{3i}x_i & \sum_l a^{3-l}_{13}y_l & \sum_l a^{3-l}_{23}y_l & \sum_l a^{3-l}_{33}y_l & z &&&&& \\
z_2 & \sum_i a^2_{2i}x_i & \sum_i a^1_{2i}x_i & \sum_l a^{3-l}_{12}y_l & \sum_l a^{3-l}_{22}y_l & \sum_l a^{3-l}_{23}y_l && z &&&& \\
z_3 & \sum_i a^2_{1i}x_i & \sum_i a^1_{1i}x_i & \sum_l a^{3-l}_{11}y_l & \sum_l a^{3-l}_{12}y_l & \sum_l a^{3-l}_{13}y_l &&& z &&& \\
z_4 &&& \sum_i a^1_{1i}x_i & \sum_i a^1_{2i}x_i & \sum_i a^1_{3i}x_i &&&& z && \\
z_5 &&& \sum_i a^2_{1i}x_i & \sum_i a^2_{2i}x_i & \sum_i a^2_{3i}x_i &&&&& z & \\
z_6 & z_5 & z_4 & z_3 & z_2 & z_1 & x_3 & x_2 & x_1 & y_2 & y_1 & z
\end{pmatrix}.$$

To prove that Strassen's equations are satisfied, take a matrix M in variables as above and a matrix M' with primed variables, then it is sufficient to show that each entry of the product matrix MM' is symmetric as a bilinear form. This is obvious for:

(1) a product of any of the first $2k$ rows of M with any column of M',
(2) a product of any of the rows $2k+1, \dots, 4k-1$ of M with any of the columns $2, \dots, 4k$ of M',

(3) a product of the last row of M with column 1 or any of the columns $2k+1,\dots,4k$ of M'.

As all matrices are symmetric it remains to check the assumption for the product of rows $2k+1,\dots,4k-1$ of M and the first column of M'.

For the row $3k+l$ we obtain the symmetric linear form $\sum_j(\sum_i a^l_{ji}x_i)x'_j$.

For the row $2k+n$ we obtain the symmetric linear form $\sum_{l=1}^{k-1}(\sum_i a^{k-l}_{k+1-n,i}x_i)y'_l+\sum_{u=1}^{k}(\sum_l a^{k-l}_{(k+1-n),u}y_l)x'_u$. □

6. Tensors of border rank m in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$

In this section we present sufficient conditions for tensors to be of minimal border rank and determine the ranks of several examples of minimal border rank.

6.1. **Centralizers of a regular element.** Let $A=B=C=\mathbb{C}^m$. An element $x\in\operatorname{End}(B)$ is *regular* if $\dim\mathbf{C}(x)=m$, where $\mathbf{C}(x):=\{y\in\operatorname{End}(B)\mid[x,y]=0\}$ is the *centralizer* of x. We say x is *regular semi-simple* if x is diagonalizable with distinct eigenvalues. Note that x is regular semi-simple if and only if $\mathbf{C}(x)$ is diagonalizable.

The following proposition was communicated to us by L. Manivel.

Proposition 6.1. *Let $U\subset\operatorname{End}(B)$ be an abelian subspace of dimension m. If there exists $x\in U$ that is regular, then U lies in the Zariski closure of the diagonalizable m-planes in $G(m,\operatorname{End}(B))$, i.e., $U\in Red(m)$. More generally, if there exist $x_1,x_2\in U$, such that U is their common centralizer, then $U\in Red(m)$.*

Proof. Since the Zariski closure of the set of regular semi-simple elements is all of $\operatorname{End}(B)$, for any $x\in\operatorname{End}(B)$, there exists a curve x_t of regular semi-simple elements with $\lim_{t\to0}x_t=x$. Consider the induced curve in the Grassmannian $\mathbf{C}(x_t)\subset G(m,\operatorname{End}(B))$. Then $\mathbf{C}_0:=\lim_{t\to0}\mathbf{C}(x_t)$ exists and is contained in $\mathbf{C}(x)\subset\operatorname{End}(B)$ and since U is abelian, we also have $U\subseteq\mathbf{C}(x)$. But if x is regular, then $\dim\mathbf{C}_0=\dim(U)=m$, so $\lim_{t\to0}\mathbf{C}(x_t)$, $\mathbf{C}_0$ and U must be equal and thus U is a limit of diagonalizable subspaces.

The proof of the second statement is similar, as a pair of commuting matrices can be approximated by a pair of diagonalizable commuting matrices and diagonalizable commuting matrices are simultaneously diagonalizable, cf. [24, Prop. 4]. □

Corollary 6.2. *Let $T\in A\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ with $\dim A\le m$ be such that $T(A^*)$ contains an element of rank m and, after using it to embed $T(A^*)\subset\mathfrak{gl}_m$, it is abelian and contains a regular element. Then $\underline{\mathbf{R}}(T)=m$.*

Proposition 6.3. *Let $T(A^*/\mathbb{C}\alpha)\subset\mathfrak{sl}(B)$ be the centralizer of a regular element of Jordan type $(d_1,\cdots,d_q)$. Then $\underline{\mathbf{R}}(T)=m$ and $\mathbf{R}(T)=2(\sum_{j=1}^q d_j)-q$. In particular, if $T(A^*)$ is the centralizer of a regular nilpotent element, then $\mathbf{R}(T)=2m-1$ and if it is the centralizer of a regular semi-simple element then $\mathbf{R}(T)=m$.*

Proof. It remains to show the second assertion. The substitution method gives the lower bound on $\mathbf{R}(T)$ for the regular nilpotent case, and Theorem 4.1 gives the lower bound for the general case. For the upper bound, it is sufficient to prove the regular nilpotent case. The tensor $M_{\mathbb{C}[\mathbb{Z}_{2m-1}]}$ (see Equation (1)) specializes to T as follows: consider the space $M_{\mathbb{C}[\mathbb{Z}_{2m-1}]}(A^*)$, cut the first $\lfloor\frac{m}{2}\rfloor$ rows and the last $\lfloor\frac{m}{2}\rfloor$ columns and set all entries appearing above the diagonal in the remaining matrix to zero. E.g., when $m=2$,

$$\begin{pmatrix}x_1&x_2&x_3\\x_3&x_1&x_2\\x_2&x_3&x_1\end{pmatrix}\to\begin{pmatrix}x_3&x_1\\x_2&x_3\end{pmatrix}\to\begin{pmatrix}x_3&0\\x_2&x_3\end{pmatrix}.$$

□

6.2. **The flag algebras of** [24]**.** We answer a question posed in [24, p. 4/p. 5] whether certain algebras derived from flags belong to $Red(m)$. We start by presenting these algebras. Using matrix notation, the algebras are given by a partition λ of size $|\lambda| = m-1$, to which we associate a Young tableau with entries $\{x_2, \cdots, x_m\}$ whose reflection (across a vertical line for the American presentation and across a diagonal line for the French presentation) we situate in the upper right hand block of the $m \times m$ matrix $T(A^*)$ and we fill the diagonal with x_1's. For example $\lambda = (4, 2, 1)$ gives rise to the following 8-dimensional subspace of $\mathbb{C}^8 \otimes \mathbb{C}^8$:

$$T(A^*) = \begin{pmatrix} x_1 & & & & x_2 & x_3 & x_4 & x_5 \\ & x_1 & & & & & x_6 & x_7 \\ & & x_1 & & & & & x_8 \\ & & & x_1 & & & & \\ & & & & x_1 & & & \\ & & & & & x_1 & & \\ & & & & & & x_1 & \\ & & & & & & & x_1 \end{pmatrix}. \tag{5}$$

Call this space the abelian Lie algebra associated to the flag induced by λ, and the corresponding tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ the tensor associated to the flag induced by λ.

Proposition 6.4. *The abelian Lie algebras associated to flags, defined in* [24, p. 4/p. 5] *belong to $Red(m)$, the closure of the diagonalizable algebras. That is, the associated tensors to these abelien Lie algebras are of border rank m.*

Proof. For the purposes of this proof, it will be convenient to re-order bases such that the x_1's occur on the anti-diagonal, and the Young tableau occur with the American presentation:

$$T(A^*) = \begin{pmatrix} x_2 & x_3 & x_4 & x_5 & & & & x_1 \\ x_6 & x_7 & & & & & x_1 & \\ x_8 & & & & & x_1 & & \\ & & & & x_1 & & & \\ & & & x_1 & & & & \\ & & x_1 & & & & & \\ & x_1 & & & & & & \\ x_1 & & & & & & & \end{pmatrix}. \tag{6}$$

Suppose that $T(A^*)$ is defined by a partition $\lambda = (k^{l_k}, \cdots, 2^{l_2}, 1^{l_1})$ with $|\lambda| = m-1$. We define m rank 1 matrices, parametrized by ϵ such that $T(A^*)$ equals the limit of the span of these matrices as $\epsilon \to 0$, considered as a curve in the Grassmannian $G(m, \mathfrak{gl}(B))$. The matrices will belong to five groups.

We label the rows and columns of our matrix by $0, \cdots, m-1$.

1) The first group contains just one matrix with four nonzero entries:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \epsilon^{m-1} \\ & & \vdots & & \\ \epsilon^{m-1} & 0 & \dots & 0 & \epsilon^{2m-2} \end{pmatrix}.$$

2) The second group also contains one matrix, with support contained in the $\ell(\lambda) \times k$ upper left rectangle. For (i, j) in the rectangle, we fill the entry with ϵ^{i+j} and set all other entries to

zero. So the tensor (6) gives

$$\begin{pmatrix} \epsilon^0 & \epsilon^1 & \epsilon^2 & \epsilon^3 & & & \\ \epsilon^1 & \epsilon^2 & \epsilon^3 & \epsilon^4 & & & \\ \epsilon^2 & \epsilon^3 & \epsilon^4 & \epsilon^5 & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}.$$

3) The third group contains $m-k-\ell(\lambda)$ matrices. Each matrix corresponds to an entry in the Young diagram λ that is neither in the zero-th row or column. Notice that the number of such entries equals the number of anti-diagonal entries of the matrix that are not in the first $k-1$ rows or first $(\sum_{i=1}^{k} l_i)-1$ columns. Fix a bijection between them. To each such entry of the Young diagram we associate a rank one matrix with only four nonzero entries. Suppose that the entry of the Young diagram is the (i_0, j_0) entry of the matrix and the corresponding entry of the anti-diagonal is $(i_1, j_1 = m-1-i_1)$. The entries are

$$a^i_j = \begin{Bmatrix} \epsilon^{i_0+j_0} & (i,j)=(i_0,j_0) \\ \epsilon^{m-1} & (i,j)=(i_1,j_1) \\ \epsilon^{i_1+j_0} & (i,j)=(i_1,j_0) \\ \epsilon^{i_0+j_1} & (i,j)=(i_0,j_1) \\ 0 & \text{otherwise} \end{Bmatrix}.$$

The tensor (6) has

$$\begin{pmatrix} & & & & \\ & \epsilon^2 & & \epsilon^5 & \\ & & & & \\ & \epsilon^4 & & \epsilon^7 & \\ & & & & \\ & & & & \end{pmatrix}.$$

4) The fourth group contains $k-1$ matrices. These correspond to the entries in the 0-th row of λ but not in the 0-th column. The matrix corresponding to the entry in the i-th column is defined by

$$a^i_j = \begin{Bmatrix} \epsilon^i & j=0 \\ \epsilon^{i+j} & j \le k-1 \text{ and } (i,j) \notin \text{Young diagram of} \lambda \\ -\epsilon^{i+j} & a^{m-1-j}_j \text{has been associated to } a^i_j \\ \epsilon^{m-1} & j=m-1-i \\ 0 & \text{otherwise} \end{Bmatrix}$$

The tensor (6) has

$$\begin{pmatrix} \epsilon^1 & & & & & & \\ & & & & & & \\ \epsilon^3 & & & & & & \\ -\epsilon^4 & & & & & & \\ & & & & & & \\ & & & & & & \\ \epsilon^7 & & & & & & \end{pmatrix}, \begin{pmatrix} & \epsilon^2 & & & & & \\ & \epsilon^3 & & & & & \\ & \epsilon^4 & & & & & \\ & & & & & & \\ & & & & & & \\ & \epsilon^7 & & & & & \\ & & & & & & \end{pmatrix}, \begin{pmatrix} & & \epsilon^3 & & & & \\ & & \epsilon^4 & & & & \\ & & \epsilon^5 & & & & \\ & & & & & & \\ & & \epsilon^7 & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}.$$

5) The fifth group, consisting of $\ell(\lambda) - 1$ matrices, is analogous to the fourth with entries corresponding to rows instead of columns and all entries, apart from the first column, in the $\ell(\lambda) \times k$ upper left rectangle equal to zero.

The tensor (6) has

$$\begin{pmatrix} \epsilon^1 & & & & -\epsilon^5 & & \epsilon^7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}, \begin{pmatrix} & & & & & & \\ \epsilon^2 & & & & \epsilon^7 & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}.$$

Except for the second group, each matrix in each group has a distinguished element in the Young diagram which, after normalization, is the limit as $\epsilon \to 0$. Moreover, summing all matrices from groups $1, 3, 4, 5$ and subtracting the matrix from group 2, the limit as $\epsilon \to 0$ is the identity matrix. $\square$

6.3. **Case** $m = 4$. Fix an A-concise, 1_A-generic border rank 4 tensor $T \in A \otimes B \otimes C$, where $A, B, C \simeq \mathbb{C}^4$. The contraction $T(A^*)$ is a 4-dimensional subspace of matrices and we may assume that it contains the identity. By [24, Prop. 18] we may assume that the space $T(A^*)$ is one of the 14 types of [24, §3.1], corresponding to orbits of PGL_4. We consider tensors up to isomorphism.

- One-regular algebras - centralizers of regular elements. There are 5 types, their ranks are provided by Proposition 6.3. These are $\mathcal{O}_{12}, \mathcal{O}_{11}, \mathcal{O}_{10'}, \mathcal{O}_{10''}, \mathcal{O}_9$ in [24, §3.1].
- Containing a $(3,1)$ Jordan type non-regular element. There are two types, giving rise to isomorphic rank 6 tensors. Set theoretically, the intersection with the Segre variety is a line and a point.

$$T_{\mathcal{O}_8'', \mathcal{O}_8'} = \begin{pmatrix} c & 0 & a & 0 \\ 0 & c & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

- Containing a nilpotent element with Jordan block size 3. There are three types, all of rank 7, the first one representing a class to which the Coppersmith-Winograd tensor $\tilde{T}_{2,CW}$ belongs, see §7.2. In the second case the intersection with the Segre set-theoretically is a line.

$$T_{\mathcal{O}_8} = \tilde{T}_{2,CW} = \begin{pmatrix} c & b & a & d \\ 0 & c & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & c \end{pmatrix}, \quad T_{\mathcal{O}_7'',\mathcal{O}_7'} = \begin{pmatrix} c & b & a & 0 \\ 0 & c & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & c \end{pmatrix},$$

- Four types, all of rank 7, giving rise to three different types of tensors. Set-theoretically the intersection with the Segre is, in the first case two lines intersecting in a point, in the second case a smooth quadric, in the third case a plane.

$$T_{\mathcal{O}_6} = \begin{pmatrix} c & 0 & a & b \\ 0 & c & 0 & d \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \quad T_{\mathcal{O}_7} = \begin{pmatrix} c & 0 & a & d \\ 0 & c & b & a \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \quad T_{\mathcal{O}_3',\mathcal{O}_3''} = \begin{pmatrix} c & a & b & d \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

6.4. **Case $m = 5$.** As remarked in [24], it is sufficient to consider nilpotent subspaces as others are built out of them, so we restrict our attention to them. Up to transpositions the following are the only maximal, nilpotent, End-abelian 5-dimensional subalgebras of the algebra of 5×5 matrices. We prove that each of them is in $Red(5)$. Notation is such that $T_{N_{i,j}}$ corresponds to the nilpotent algebras N_i, N_j of [44], and we slightly abuse notation, identifying the tensor with its corresponding linear space.

$$T_{N_{1,4}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & 0 & a & 0 & 0 \\ d & 0 & 0 & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{6,8}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & e & 0 \\ c & b & a & d & e \\ 0 & 0 & 0 & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{7,9}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & d & 0 \\ c & b & a & e & d \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & b & a \end{pmatrix},$$

$$T_{N_{10,12}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ d & 0 & 0 & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{11,13}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & d & 0 \\ d & 0 & 0 & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{14}} = \tilde{T}_{3,CW} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & d & e \\ d & 0 & 0 & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix},$$

$$T_{N_{15}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ d & c & b & a & 0 \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{16}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ d & c & b & a & e \\ e & 0 & 0 & 0 & a \end{pmatrix}, \quad T_{N_{17}} = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ d & c & b & a & e \\ 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

$T_{N_{1,4}}$ is obviously of border rank five. For $T_{N_{6,8}}$,$T_{N_{15}}$,$T_{N_{16}}$, and $T_{N_{17}}$ (resp. $T_{N_{7,9}}$) apply Proposition 6.1 to a pair of matrices represented by b and e (resp. b,d). $T_{N_{10,12}}$ is the limit as $\epsilon \to 0$ of the space spanned by the following five matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \epsilon & \epsilon^2 & 0 & 0 & 0 \\ 1 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^2 & -\epsilon^5 & \epsilon^6 & 0 & 0 \\ -\epsilon & \epsilon^4 & -\epsilon^5 & 0 & 0 \\ \epsilon^{-2} & -\epsilon & \epsilon^2 & 0 & 0 \\ -1 & \epsilon^3 & -\epsilon^4 & 0 & 0 \\ -1 & \epsilon^3 & -\epsilon^4 & 0 & 0 \end{pmatrix}.$$

$T_{N_{11,13}}$ is the limit of the space spanned by

$$\begin{pmatrix} 0&0&0&0&0\\ \epsilon&\epsilon^2&0&0&0\\ 1&\epsilon&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ \epsilon^{-3}&0&0&1&0\\ 1&0&0&\epsilon^3&0\\ 0&0&0&0&0\end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&-\epsilon^2&\epsilon^2\end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}, \begin{pmatrix} \epsilon^2&-\epsilon^5&\epsilon^6&-\epsilon^4&0\\ -\epsilon&\epsilon^4&-\epsilon^5&\epsilon^3&0\\ \epsilon^{-2}&-\epsilon&\epsilon^2&-1&0\\ -1&\epsilon^3&-\epsilon^4&\epsilon^2&0\\ -1&\epsilon^3&-\epsilon^4&\epsilon^2&0\end{pmatrix}.$$

$T_{N_{14}}$ is isomorphic to the Coppersmith-Winograd tensor. We determine the rank of each tensor.

The following conjecture was presented at the 2011 *Algebraic geometry with a view to applications* semester at the Mittag-Leffler institute:

Conjecture 6.5 (J. Rhodes)**.** [4, Conjecture 0] *The maximal rank of a tensor in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ of border rank m is $2m-1$.*

This conjecture was known to be true classically for $m=1,2$ and verified in [4] for $m=3$. Above we verified it for $m=4$. The following proposition shows that the conjecture is false for $m>4$:

Proposition 6.6.

$$\mathbf{R}(T_{N_{1,4}})=\mathbf{R}(T_{N_{10,12}})=\mathbf{R}(T_{N_{11,13}})=\mathbf{R}(T_{N_{14}})=\mathbf{R}(T_{N_{15}})=\mathbf{R}(T_{N_{16}})=\mathbf{R}(T_{N_{1,4}})=9$$

and

$$\mathbf{R}(T_{N_{6,8}})=\mathbf{R}(T_{N_{7,9}})=10.$$

Proof. The fact that the rank of any tensor is at least 9 follows by the substitution method. To see that $\mathbf{R}(T_{N_{6,8}})\geq 10$, first apply Proposition 3.1 to the first and fourth row and to the third and fifth column. This shows that the rank is at least 4 plus the rank of the tensor associated to

$$\begin{pmatrix} b&0&e\\ c+\alpha e&b+\beta e&d+\gamma e\\ e&0&\end{pmatrix},$$

where α,β,γ are some constants. Now apply the proposition to the second column obtaining a tensor represented by

$$\begin{pmatrix} b&e\\ c+\alpha e+\delta b&d+\gamma e+\rho b\\ e&0\end{pmatrix},$$

where $\delta,\rho,\alpha,\gamma$ are (possibly new) constants. This space is equal to

$$\begin{pmatrix} b&e\\ c&d\\ e&0\end{pmatrix}$$

and it remains to show that it corresponds to a tensor of rank at least 5. This follows by Proposition 3.1 by first reducing b,c,d and obtaining a rank 2 matrix.

To prove that $R(T_{N_{7,9}})\geq 10$, apply Proposition 3.1 and remove the second, third and fifth column and first and fourth row to obtain a tensor isomorphic to

$$\begin{pmatrix} b&d\\ c&e\\ 0&b\end{pmatrix},$$

and conclude as above.

The upper bounds for ranks of $T_{N_{1,4}}$, $T_{N_{10,12}}$, $T_{N_{15}}$ and $T_{N_{17}}$ follow from Proposition 6.3.

For $T_{N_{6,8}}$, consider:

(1) 5 matrices, including the matrix corresponding to c, which follow from Proposition 6.3 for the upper left 3×3 corner,
(2) 2 matrices for last 2 diagonal entries,
(3) 1 matrix corresponding to d,
(4) the 2 matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $T_{N_{11,13}}$ it is enough to notice that once a matrix for c and the fourth diagonal entry are given, one can generate the matrix corresponding to d using

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

An analogous method shows $\mathbf{R}(T_{N_{16}}) = 9$. The tensor $T_{N_{14}}$ is a special case of Proposition 7.1.

For $T_{N_{7,9}}$, consider 3 rank one matrices corresponding to entries of the first column, 2 rank one matrices corresponding to the third and fourth diagonal entry and one matrix corresponding to e. Apart from these six matrices we are left with the tensor represented by

$$\begin{pmatrix} a & d & 0 \\ b & 0 & d \\ 0 & b & a \end{pmatrix}.$$

This tensor is isomorphic to the symmetric tensor given by the monomial xyz which has Waring rank 4, see, e.g., [31] (the upper bound dates back at least to [18]), and thus tensor rank at most 4. □

Apart from the nilpotent algebras just discussed there are two families of End-closed 5-dimensional subalgebras.

(1) The subspace spanned by identity and any 4-dimensional subspace of the 6-dimensional algebra

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \end{pmatrix}.$$

In this case there are normal forms: Tensors in $A' \otimes B' \otimes C' = \mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ are classified in [10]. We have $T = a_1(b_1c_1 + .. + b_5c_5) + a_2b_1c_4 + a_3b_1c_5 + a_4b_2c_4 + a_5b_2c_5 + a_6b_3c_4 + a_7b_3c_5 = a_1(b_1c_1 + .. + b_5c_5) + T'$ where $a_2, ..a_7$ satisfy two linear relations. If we make a change of basis in c_4, c_5, say by a 2×2 matrix X, then as long as we change b_4, b_5 by X^{-1} the first term does not change. Similarly, if we make a change of basis in b_1, b_2, b_3, by a matrix Y, then as long as we change c_1, c_2, c_3 by Y^{-1}, the first term does not change. In our case we may assume the tensors are A-concise. There are the following cases (numbers as in [10]), we abuse notation, writing T for $T(A^*)$:

$$T_9 = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_3 && x_1 & \\ x_4 & x_5 &&& x_1 \end{pmatrix}, \quad T_{19} = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_3 & x_5 & x_1 & \\ x_3 & x_4 &&& x_1 \end{pmatrix} T_{20} = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_4 & x_5 & x_1 & \\ x_3 &&&& x_1 \end{pmatrix},$$

$$T_{21} = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_4 & x_5 & x_1 & \\ x_3 & x_5 &&& x_1 \end{pmatrix}, T_{22} = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_4 && x_1 & \\ x_3 && x_5 && x_1 \end{pmatrix}, \quad T_{23} = \begin{pmatrix} x_1 &&&& \\ & x_1 &&& \\ && x_1 && \\ x_2 & x_3 & x_4 & x_1 & \\ x_3 & x_4 & x_5 && x_1 \end{pmatrix}.$$

Now $\underline{\mathbf{R}}(T_9) = \underline{\mathbf{R}}(T_{20}) = 5$ because they are special cases of flag-algebra tensors - cf. Proposition 6.4.

$T_{22}(A^*)$ is the limit of the space spanned by the following 5 matrices:

$$\begin{pmatrix} \epsilon^2 && -\epsilon^3 && \epsilon^4 \\ &&&& \\ &&&& \\ &&&& \\ 1 && -\epsilon && \epsilon^2 \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ 1 & -\epsilon && \epsilon^2 & \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ & \epsilon^2 &&& \\ &&&& \\ & \epsilon &&& \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ && \epsilon^2 && \\ &&&& \\ && \epsilon && \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ 1 &&&& \\ 1 &&&& \end{pmatrix}.$$

$T_{23}(A^*)$ is the limit of the space spanned by the following 5 matrices:

$$\begin{pmatrix} &&&& \\ &&&& \\ & -2\epsilon^6 & \epsilon^4 && \epsilon^8 \\ &&&& \\ & -2\epsilon^2 & 1 && \epsilon^4 \end{pmatrix}, \begin{pmatrix} \epsilon^4 &&& -\frac{1}{2}\epsilon^8 & \\ -\epsilon^3 &&& \frac{1}{2}\epsilon^7 & \\ &&&& \\ -2 &&& \epsilon^4 & \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ \epsilon^3 & \epsilon^4 & 2\epsilon^5 && \\ &&&& \\ 1 & \epsilon & 2\epsilon^2 && \\ \epsilon & \epsilon^2 & 2\epsilon^3 && \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ && 2\epsilon^2 && \\ && 1+2\epsilon^3 && \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ 1 & -\epsilon &&& \\ -\epsilon & \epsilon^2 &&& \end{pmatrix}.$$

$T_{21}(A^*)$ is the limit of the space spanned by the following 5 matrices:

$$\begin{pmatrix} &&&& \\ &&&& \\ & \epsilon^3 & \epsilon^4 && \\ & \epsilon & \epsilon^2 && \\ & \epsilon^2 & \epsilon^3 && \end{pmatrix}, \begin{pmatrix} &&&& \\ && -\epsilon^4 && \\ && \epsilon^3 && \\ && \epsilon && \\ &&&& \end{pmatrix}, \begin{pmatrix} \epsilon^4 && -\epsilon^6 & \epsilon^8 & \\ &&&& \\ &&&& \\ 1 && -\epsilon^2 & \epsilon^4 & \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ &&&& \\ \epsilon & -\epsilon^2 & -\epsilon^3 && \epsilon^4 \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ 1 &&&& \\ \epsilon &&&& \end{pmatrix}.$$

$T_{19}(A^*)$ is the limit of the space spanned by the following 5 matrices:

$$\begin{pmatrix} \epsilon^2 && -\epsilon^3 & \epsilon^4 & \\ &&&& \\ &&&& \\ 1 && -\epsilon & \epsilon^2 & \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ && \epsilon^2 && \\ && \epsilon && \\ &&&& \end{pmatrix}, \begin{pmatrix} &&&& \\ & \epsilon^2 &&& \epsilon^4 \\ &&&& \\ &&&& \\ & 1 &&& \epsilon^2 \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ 1 & 1 &&& \\ 1 & 1 &&& \end{pmatrix}, \begin{pmatrix} &&&& \\ &&&& \\ &&&& \\ -1 & 1 &&& \\ 1 & -1 &&& \end{pmatrix}.$$

Proposition 6.7. $\mathbf{R}(T_{21}) = 10$ *and all other tensors on this list have* $\mathbf{R}(T_j) = 9$.

Proof. This follows by the substitution-method and considering the ranks of T' in [10]. □

(2) The subspace spanned by the identity and any 4-dimensional subspace of the 5-dimensional algebra

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ b & a & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & c & 0 \end{pmatrix}. \tag{7}$$

All the operations we will perform preserve the identity matrix.

First, by exchanging rows and columns the algebra is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & c & 0 & 0 & 0 \\ b & e & a & 0 & 0 \end{pmatrix}.$$

Note that the tensor $T_{Leit,5}$ of Proposition 5.1 is in this family (set $b = 0$).

Proposition 6.8. *All End-closed tensors in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ obtained from 4-dimensional subspaces of (7) have border rank five.*

To prove the proposition, we will use the following lemma:

Lemma 6.9. *Each of the tensors in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ corresponding to linear spaces spanned by the identity and the following subspaces have border rank 5:*

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \\ b & e & a & 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & a & 0 & 0 & 0 \\ b & e & a & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & e & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 \end{pmatrix}, S_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & e & 0 & 0 & 0 \\ b & d & a & 0 & 0 \end{pmatrix}.$$

The tensors corresponding to S_1, S_3, S_4, have rank 9 and the tensor corresponding to S_2 has rank 10.

Proof. We first prove the statement about the border rank. The span of S_1 and the identity is the limit of the space spanned by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & 0 & \frac{1}{2}\epsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & \epsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 & -\frac{1}{2}\epsilon^3 & 0 & \frac{1}{2}\epsilon^4 \\ 0 & 0 & 0 & 0 & 0 \\ -\epsilon & 0 & \frac{1}{2}\epsilon^2 & 0 & -\frac{1}{2}\epsilon^3 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -\epsilon & 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & \frac{\epsilon^2}{4} & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & \frac{\epsilon^2}{4} & 0 & 0 & 0 \\ 4 & \epsilon & 0 & 0 & 0 \end{pmatrix}.$$

For S_2, consider

$$X_1 = \begin{pmatrix} \epsilon^4 & 0 & \epsilon^6 & 0 & \epsilon^8 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon^2 & 0 & \epsilon^4 & 0 & \epsilon^6 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \epsilon^2 & 0 & \epsilon^4 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha\beta\epsilon^3 & \beta\epsilon^4 & 0 & \alpha\beta^2\epsilon^6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & \alpha^{-1}\epsilon^2 & 0 & \beta\epsilon^4 & 0 \\ \alpha & \epsilon & 0 & \alpha\beta\epsilon^3 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & \frac{1-\alpha}{\alpha}\epsilon^2 & 0 & 0 & 0 \\ 1+\alpha & \epsilon & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{\beta(\alpha-1)}\epsilon & \epsilon^2 & \alpha\epsilon^3 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha\epsilon^3 & 0 & 0 & 0 & 0 \\ \epsilon^2 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{\beta(\alpha-1)}\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where in order that X_3 has rank one, take α to be a solution of the equation $\alpha^2 + \alpha - 1 = 0$. We determine β later.

First note that the limits as ϵ goes to zero of

$$\frac{1}{\epsilon}X_5, X_1, \frac{1}{\epsilon}X_4$$

give matrices corresponding respectively to d, b, e.

Consider $X_1 + X_2 - X_3$. The constant terms and the terms of order ϵ add to zero. The $(4,2)$ entry equals

$$(\frac{1}{\alpha} - \frac{1-\alpha}{\alpha})\epsilon^2 = \epsilon^2.$$

Hence, the limit gives the matrix corresponding to a.

It remains to prove that the identity matrix belongs to the limit. For this we consider

$$X_1 + \frac{1}{\beta}(X_2 - \frac{1}{1-\alpha}X_3) - X_4 - X_5.$$

As ϵ^4 is on the diagonal it remains to prove that the lower order terms all add to zero. For the constant term

$$1 + \frac{\alpha}{\beta} - \frac{1+\alpha}{\beta(1-\alpha)} = \frac{\beta(1-\alpha) + \alpha - \alpha^2 - 1 - \alpha}{\beta(1-\alpha)},$$

the numerator equals $\beta(1-\alpha) - \alpha^2 - 1$, so we take $\beta = \frac{\alpha^2+1}{1-\alpha}$ to make it zero. To see that the terms proportional to ϵ cancel, observe that

$$\frac{1}{\beta}(1 - \frac{1}{1-\alpha}) = -\frac{\alpha}{\beta(\alpha-1)}.$$

All three of the terms proportional to ϵ^2 cancel, the only nontrivial being

$$\frac{1}{\alpha} - \frac{1}{1-\alpha}\frac{1-\alpha}{\alpha}.$$

The term proportional to ϵ^3 also cancels out.

The span of S_3 and the identity is the limit of the space spanned by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\epsilon & 0 & \frac{1}{2}\epsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -\epsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 & \frac{1}{2}\epsilon^3 & 0 & \frac{1}{2}\epsilon^4 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon & 0 & \frac{1}{2}\epsilon^2 & 0 & \frac{1}{2}\epsilon^3 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & \epsilon & 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -\epsilon & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The span of S_4 and the identity is the limit of the space spanned by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon^6 & \epsilon^8 & 0 & 0 & \epsilon^{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon^2 & \epsilon^4 & 0 & 0 & \epsilon^8 \\ 1 & \epsilon^2 & 0 & 0 & \epsilon^6 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \epsilon^3 & 0 & \epsilon^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \epsilon^3 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^6 & 0 & -\epsilon^8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon^4 & 0 & \epsilon^6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\epsilon^3 & \epsilon^8 & \epsilon^9 & 0 & 0 \\ -\epsilon^2 & \epsilon^7 & \epsilon^8 & 0 & 0 \\ \epsilon^{-3} & -\epsilon^2 & -\epsilon^3 & 0 & 0 \end{pmatrix}.$$

The lower bounds for rank follow by substitution method. For rank upper bounds, consider the seven rank one matrices:

(1) 4 matrices corresponding to first two and last two entries of the diagonal,
(2) 1 matrix corresponding to b,
(3) the 2 matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In all four cases it is easy to find the remaining rank 1 matrices. □

Proof of Proposition 6.8. If the subspace is given by $a = 0$ then we conclude by Proposition 6.4. Otherwise there exists a matrix M_1 in the algebra with the entries corresponding to a nonzero. We may assume that the 3 other generators of the algebra M_2, M_3, M_4 have entries corresponding to a equal to zero. Further, we may assume that M_2 has only one entry nonzero, corresponding to b, as otherwise, by considering M_1^2 the algebra would not be End-closed. Hence we may assume that M_3 and M_4 have only nonzero entries on d, c and e.

Let $\tilde{S}$ denote the 3-dimensional vector space corresponding to d, c and e, and let $S \subset \tilde{S}$ be the two-dimensional subspace spanned by M_3 and M_4.

Case 1) $S = \{M \in \tilde{S} \mid c = 0\}$. We may assume that M_3 corresponds to e, M_4 corresponds to d and the algebra is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & \lambda a & 0 & 0 & 0 \\ b & e & a & 0 & 0 \end{pmatrix},$$

for some constant λ. If $\lambda = 0$ we are in case S_1. Otherwise, by multiplying second column by $1/\lambda$ and second row by λ we may assume that $\lambda = 1$ and are in case S_2.

Case 2) $S = \{M \in \tilde{S} \mid e = 0\}$. Subtract any multiple of the third column from the second column, and add the same multiplicity of the second row to the third row to reduce to case S_3.

Case 3) $S = \{M \in \tilde{S} \mid d = 0\}$. This is analogous to Case 2).

Case 4) As we are not in case 2) or 3) we may assume there are constants $\lambda \in \mathbb{C}$, and $\delta \neq 0$ such that

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d & \lambda d + \delta e & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \end{pmatrix}.$$

Then we may assume that M_1 represents the space

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \rho a & a & 0 & 0 \end{pmatrix}.$$

Subtract ρ times the third column from the second column, and add ρ times the second row to third row, reducing to $\rho = 0$. At this point we have the algebra

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & \lambda d + \delta e & 0 & 0 & 0 \\ b & e & a & 0 & 0 \end{pmatrix}.$$

Subtract $1/\delta$ times the fourth row from the fifth row and add $1/\delta$ times the fifth column to the fourth column to obtain a subspace isomorphic to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ d & e & 0 & 0 & 0 \\ b & -\frac{\lambda}{\delta} d & a & 0 & 0 \end{pmatrix}.$$

If $\lambda = 0$ we are in Case 2). If $\lambda \neq 0$ we multiply the second column by $-\frac{\delta}{\lambda}$ and the second row by $-\frac{\lambda}{\delta}$ to reduce to case S_4. $\square$

6.5. **Examples with large gaps between rank and border rank.** First consider

$$T_{gap} = a_1 \otimes (b_1 \otimes c_1 + \cdots + b_6 \otimes c_6) + a_2 \otimes b_6 \otimes c_1 + a_3 \otimes (b_5 \otimes c_1 + b_6 \otimes c_3) + a_4 \otimes b_5 \otimes c_2 + a_5 \otimes b_4 \otimes c_3 + a_6 \otimes (b_4 \otimes c_2 + b_5 \otimes c_3),$$

i.e.,

$$T_{gap}(A^*) = \begin{pmatrix} x_1 & & & & & \\ & x_1 & & & & \\ & & x_1 & & & \\ & x_6 & x_5 & x_1 & & \\ x_3 & x_4 & x_6 & & x_1 & \\ x_2 & & x_3 & & & x_1 \end{pmatrix}.$$

Note T_{gap} is neither 1_B nor 1_C-generic. In the following proposition we show that T_{gap} is also a counterexample to [4, Conjecture 0].

Proposition 6.10. $\underline{\mathbf{R}}(T_{gap}) = 6$ *and* $\mathbf{R}(T_{gap}) = 12$.

Proof. To prove that $\mathbf{R}(T_{gap}) = 12$, by the substitution method, it suffices to prove that $\mathbf{R}(T') = 6$, where T' corresponds to the subspace

$$\begin{pmatrix} & x_6 & x_5 \\ x_3 & x_4 & x_6 \\ x_2 & & x_3 \end{pmatrix}.$$

This space is contained in the space spanned by the following 6 rank one matrices:

(1) 3 matrices corresponding to x_2, x_4, x_5,
(2) 2 matrices corresponding to x_6,
(3) the matrix

$$\begin{pmatrix} & & \\ 1 & & 1 \\ 1 & & 1 \end{pmatrix}.$$

Were T' of rank 5, then the space would be equal to the space generated by five rank one matrices. However, each rank one matrix in this space has $x_3 = 0$. This finishes the proof that $\mathbf{R}(T_{gap}) = 12$. It remains to prove $\underline{\mathbf{R}}(T_{gap}) = 6$. As T_{gap} is A-concise we only need to prove $\underline{\mathbf{R}}(T_{gap}) \leq 6$. Consider the following 6 rank one matrices:

$$\begin{pmatrix} &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \\ 1 &&&&& \end{pmatrix}, \begin{pmatrix} &&&&& \\ &&&&& \\ &&&&& \\ &&1&&& \\ &&&&& \\ &&&&& \end{pmatrix}, \begin{pmatrix} &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \\ -\epsilon^4 & -1 &&&& \\ &&&&& \end{pmatrix},$$

$$\begin{pmatrix} \epsilon^5 && \epsilon^9 &&& \epsilon^{10} \\ &&&&& \\ &&&&& \\ &&&&& \\ \epsilon^4 && \epsilon^8 &&& \epsilon^9 \\ 1 && \epsilon^4 &&& \epsilon^5 \end{pmatrix}, \begin{pmatrix} &&&&& \\ & \epsilon^5 & \epsilon^8 && \epsilon^{10} & \\ &&&&& \\ & \epsilon^3 & \epsilon^6 && \epsilon^8 & \\ & 1 & \epsilon^3 && \epsilon^5 & \\ &&&&& \end{pmatrix}, \begin{pmatrix} &&&&& \\ &&&&& \\ & \epsilon^8 & \epsilon^5 & -\epsilon^{10} && \\ & -\epsilon^3 & -1 & \epsilon^5 && \\ & -\epsilon^6 & -\epsilon^3 & \epsilon^8 && \\ & -\epsilon^7 & -\epsilon^4 & \epsilon^9 && \end{pmatrix}.$$

□

So far all the tensors we considered had rank less than or equal to twice the border rank. By [7] the maximal rank of a tensor is at most twice the maximal border rank, and all previously known examples of tensors had rank at most twice the border rank. The following example goes beyond this ratio.

For $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, consider $A \otimes A'$ as a single vector space and similarly for $B \otimes B'$ and $C \otimes C'$. Define $T \otimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$. In what follows, we will use a tensor $T' \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to produce three copies of T_{gap}.

Proposition 6.11. *The tensor*

$$T_{biggap} := T_{gap} \otimes (e_1 \otimes f_1 \otimes g_1 + e_2 \otimes (f_2 \otimes g_1 + f_1 \otimes g_2)) \in (\mathbb{C}^{12})^{\otimes 3}$$

has border rank 12 *and rank at least* 25.

Proof. T_{biggap} has border rank 12 as it is the tensor product of concise tensors of border rank 6 and 2. In terms of matrices,

$$T_{biggap}(A^* \otimes \mathbb{C}^{2*}) = \begin{pmatrix} T_{gap}(A^*) & T_{gap}(\tilde{A}^*) \\ T_{gap}(\tilde{A}^*) & \end{pmatrix}$$

where $\tilde{A}^*$ is another copy of A^*. Denote the vectors in $T(\tilde{A}^*)$ with primes. Eliminate x_1, x_1' by the substitution method so that T_{biggap} has border rank at least 9 plus the rank of the tensor $\tilde{T} \in \mathbb{C}^6 \otimes \mathbb{C}^6 \otimes \mathbb{C}^{10}$ represented by

$$\begin{pmatrix} 0 & x_6 & x_5 & 0 & x_6' & x_5' \\ x_3 & x_4 & x_6 & x_3' & x_4' & x_6' \\ x_2 & 0 & x_3 & x_2' & 0 & x_3' \\ 0 & x_6' & x_5' & 0 & 0 & 0 \\ x_3' & x_4' & x_6' & 0 & 0 & 0 \\ x_2' & 0 & x_3' & 0 & 0 & 0 \end{pmatrix}.$$

We now prove $\mathbf{R}(\tilde{T}) \geq 13$. Write $\tilde{T} = x_2 \otimes M_2 + \cdots + x_6' \otimes M_6'$. Apply Proposition 3.1 first with x_2 to get a tensor $\tilde{T}^{(1)} = x_3 \otimes (M_3 - \lambda_3 M_2) + \cdots + x_6' \otimes (M_6' - \lambda_6' M_2)$ with $\mathbf{R}(\tilde{T}^{(1)}) \leq \mathbf{R}(\tilde{T}) - 1$. Continue in this manner, eliminating all but x_3' to get a tensor $\tilde{T}^{(9)} = x_3 \otimes (M_3 + c_2 M_2 + \cdots + c_6' M_6') \in \mathbb{C}^1 \otimes \mathbb{C}^6 \otimes \mathbb{C}^6$ where the c_j, c_j' are some constants.

Hence $\mathbf{R}(\tilde{T}) \geq 9 + \mathbf{R}(\tilde{T}^{(9)})$. But $\mathbf{R}(\tilde{T}^{(9)})$ is simply the (usual) rank of the matrix

$$\tilde{M}_3 = \begin{pmatrix} 0 & c_6 & c_5 & 0 & c_6' & c_5' \\ c_3 & c_4 & c_6 & 1 & c_4' & c_6' \\ c_2 & 0 & c_3 & c_2' & 0 & 1 \\ 0 & c_6' & c_5' & 0 & 0 & 0 \\ 1 & c_4' & c_6' & 0 & 0 & 0 \\ c_2' & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It remains to show that $\operatorname{rank}(\tilde{M}_3) \geq 4$. Suppose to the contrary that all the 4×4 minors of $\tilde{M}_3$ are zero. One of the minors equals $(1 - c_2' c_6')^2$, hence we would need $c_2', c_6' \neq 0$. There is also a minor $(c_2' c_5')^2$, which would force $c_5' = 0$. However, there is also the minor $(c_4' c_5' - c_6'^2)^2$ which under these assumptions cannot be zero. We conclude $\mathbf{R}(T) \geq 12 + 9 + 4 = 25$. □

By further tensoring T_{gap} analogously as above, we obtain tensors with rank to border rank ratio converging at least to 13/6.

The following tensor is a generalization of S_2 of Lemma 6.9, which is the case $T_{biggap,5}$.

Theorem 6.12. *Let $m = 2k + 1$. Let $\tilde{T} \in \mathbb{C}^{m-1} \otimes \mathbb{C}^{k+1} \otimes \mathbb{C}^{k+1}$ be the tensor represented by*

$$\begin{pmatrix} x_0 & 0 & 0 & \dots & 0 & 0 \\ x_1 & x_0 & 0 & \dots & 0 & 0 \\ x_2 & 0 & x_0 & \vdots & 0 & 0 \\ \vdots & & & & & \\ x_{k-1} & 0 & 0 & \dots & x_0 & 0 \\ x_k & x_{k+1} & x_{k+2} & \dots & x_{2k-1} & x_0 \end{pmatrix}.$$

Let $T_{biggap,m} \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be the tensor such that $T_{biggap,m}(A^*)$ has $\tilde{T}(\mathbb{C}^{m-1*})$ in its lower left $(k+1) \times (k+1)$ corner and x_{2k} along the diagonal. Then $m \leq \underline{\mathbf{R}}(T_{biggap,m}) \leq m+1$ and $\mathbf{R}(T_{biggap,m}) = \frac{5}{2}(m-1)$. In particular the ratio of rank to border rank tends to $\frac{5}{2}$ as $m \to \infty$.

Proof. It is straightforward that $\mathbf{R}(\tilde{T}) = 3k$. By the substitution method $\mathbf{R}(T_{biggap,m}) \geq 2k + \mathbf{R}(\tilde{T}) = 5k$ and in fact equality holds.

To estimate the border rank, fix bases $\{a_j\}$ of A, $\{b_j\}$ of B and $\{c_j\}$ of C so that $\tilde{T}$ belongs to the subspace $\operatorname{span}\{a_1, \dots, a_{m-1}\} \otimes \operatorname{span}\{b_{k+1}, \dots, b_m\} \otimes \operatorname{span}\{e_1, \dots, e_{k+1}\}$. Consider the following m rank 1 elements of $B \otimes C$:

(1) $(2b_m + \epsilon b_{k+1} + \epsilon^2 b_1) \otimes (c_1 + \frac{1}{2}\epsilon c_{k+1} + \frac{1}{2}\epsilon^2 c_m)$,

(2) $(2b_m - \epsilon b_{k+1} + \epsilon^2 b_1)\otimes(c_1 - \frac{1}{2}\epsilon c_{k+1} + \frac{1}{2}\epsilon^2 c_m)$,
(3) $(b_{k+1} + \cdots + b_{m-1} + 2b_m)\otimes c_1$,
(4) $b_m\otimes(2c_1 + c_2 + \cdots + c_k)$,
(5) $b_i\otimes(c_1 + \epsilon c_{i-k+1} + \epsilon^2 c_i)$ for $i = k+2, \dots, m-1$,
(6) $(b_m - \epsilon b_{k+i} + \epsilon^2 b_i)\otimes c_i$ for $i = 2, \dots, k$.

The rank one elements of $\tilde{T}(\mathbb{C}^{m-1*})$ are obtained from 5) and 6). The diagonal of $\tilde{T}(\mathbb{C}^{m-1*})$ is obtained by adding all elements of 5) with 1) and subtracting 3). The identity matrix is obtained by adding all elements of 5) and 6) with 1) and 2) and subtracting 3) and 4). □

Remark 6.13. After we posted a preprint of this article on the arXiv, Jeroen Zuiddam shared with us his forthcoming article (now [48]) presenting an example of a sequence of tensors with rank to border rank ratio approaching three.

Question 6.14. Is the ratio of rank to border rank unbounded? Can one find explicit tensors with ratio 3 or larger?

In this context we recall the following problem, which is a variant of our question in the situation of minimal border rank:

Problem 6.15. [6, Open problem 4.1] Is there an explicit family of tensors $T_m \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ with $\mathbf{R}(T_m) \geq (3+\epsilon)m$ for some $\epsilon > 0$? Can we even achieve this for tensors corresponding to the multiplication in an algebra, i.e., is there an explicit family of algebras $\mathcal{A}_m$ with $\mathbf{R}(\mathcal{A}_m) \geq (3+\epsilon)\dim\mathcal{A}_m$ for some $\epsilon > 0$?

For tensors $T \in A_1\otimes\cdots\otimes A_n$, there are tensors of rank $n-1$ of border rank two.

6.6. **There are parameters worth of non-isomorphic 1-generic border rank m tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$.** Let $\tau \in Mat_{p\times n}$. Set $m = n+p+1$. Define

$$T_{Leit,\tau} := a_1\otimes(b_1\otimes c_1 + \cdots\otimes b_{p+n+1}\otimes c_{p+n+1}) + \sum_{j=1}^{p} a_{1+j}\otimes b_j\otimes(\sum_{s=1}^{n}\tau_{j,s}c_{p+1+s}) + \sum_{s=1}^{n} a_{p+1+s}b_{p+1}c_{p+1+s}$$

Leitner [33] shows that $\underline{\mathbf{R}}(T_{Leit,\tau}) = m$ and that the family gives non-isomorphic tensors for $p \geq 4$, $n \geq 2$, i.e., $m \geq 7$.

Remark 6.16. Leitner only shows the border rank condition under certain genericity hypotheses on τ, but from the border rank perspective they are unnecessary by taking limits. (Border rank is semi-continuous.)

In particular, when $p = 4, n = 2$ Leitner shows that there is a one-parameter family of non-isomorphic subgroups of $SL_n\mathbb{R}$ that are limits under conjugacy of the torus. The same argument shows that there is a corresponding one-parameter family of non-isomorphic tensors.

7. Coppersmith-Winograd value

As mentioned in the introduction, a motivation for this article is the study of upper bounds for the exponent of matrix multiplication. For our purposes, the *exponent* ω of matrix multiplication, which governs the complexity of the matrix multiplication tensor $M_{\langle \mathbf{n}\rangle} \in \mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}$, may be defined as

$$\omega := \inf\{\tau \in \mathbb{R} \mid \underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) = O(\mathbf{n}^\tau)\}.$$

Naïvely one has $\omega \leq 3$ and it is generally conjectured by computer scientists that $\omega = 2$.

The "proper" way to determine the exponent of matrix multiplication would be to determine the border rank of the matrix multiplication tensor. Unfortunately, this appears to be beyond our current capabilities. Thanks to a considerable amount of work, most notably [38, 41, 16], one can prove upper bounds for matrix multiplication by considering other tensors.

First, Schönhage's asymptotic sum inequality [38] states that for all $\mathbf{l}_i, \mathbf{m}_i, \mathbf{n}_i$, with $1 \le i \le s$,

$$\sum_{i=1}^{s} (\mathbf{m}_i \mathbf{n}_i \mathbf{l}_i)^{\frac{\omega}{3}} \le \underline{\mathbf{R}}(\bigoplus_{i=1}^{s} M_{\langle \mathbf{m}_i, \mathbf{n}_i, \mathbf{l}_i \rangle}).$$

Then Strassen [41] pointed out that it would be sufficient to find upper bounds on the border rank of a tensor that degenerated into a disjoint sum of matrix multiplication tensors. This was exploited most successfully by Coppersmith and Winograd [16], who attained their success with a tensor $\tilde{T}_{CW}$. The purpose of this section is to isolate geometric aspects of this tensor in the hope of finding other tensors that would enable further upper bounds on the exponent.

In practice, only tensors of minimal, or near minimal border rank have been used to prove upper bounds on the exponent. Call a tensor T that gives a "good" upper bound for the exponent via the methods of [41, 16], of *high Coppersmith-Winograd value* or *high CW-value* for short. More precisely, T has high Coppersmith-Winograd value if the quantity $Val_\rho(T^{\otimes k})$ as defined in [2, p. 8] is large for some k. We briefly review tensors that have been utilized. Our study is incomplete because the CW-value of a tensor also depends on its presentation, and in different bases a tensor can have quite different CW-values. Moreover, even determining the value in a given presentation still involves some "art" in the choice of a good decomposition, choosing the correct tensor power, estimating the value and probability of each block [46].

7.1. **Schönhage's tensors.** Schönhage's tensors are $T_{Sch} = M_{\langle N,1,1\rangle} \otimes M_{\langle 1,m,n\rangle}$ where $N = (m-1)(n-1)$. Here $\underline{\mathbf{R}}(T_{Sch}) = N+1$ while $\mathbf{R}(T_{Sch}) = N+mn = 2\underline{\mathbf{R}}(T_{Sch}) - (m+n-1)$. It gives $\omega < 2.55$. There is nothing to gain by taking tensor powers here because the two matrix multiplications are already disjoint.

7.2. **The Coppersmith-Winograd tensors.** Coppersmith and Winograd define two tensors:

$$T_{q,CW} := \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0 \in \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \tag{8}$$

and

$$\tilde{T}_{q,CW} := \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) + a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0 \in \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2} \tag{9}$$

both of which have border rank $q+2$.

In terms of matrices,

$$T_{q,CW}(C^*) = \begin{pmatrix} 0 & x_1 & \cdots & & x_q \\ x_1 & x_0 & 0 & \cdots & \\ x_2 & 0 & x_0 & & \\ \vdots & \vdots & & \ddots & \\ x_q & 0 & \cdots & 0 & x_0 \end{pmatrix}$$

and

$$\tilde{T}_{q,CW}(C^*) = \begin{pmatrix} x_{q+1} & x_1 & \cdots & & x_q & x_0 \\ x_1 & x_0 & 0 & \cdots & & 0 \\ x_2 & 0 & x_0 & & & \\ \vdots & \vdots & & \ddots & & \\ x_q & 0 & \cdots & 0 & x_0 & \\ x_0 & 0 & \cdots & & 0 & 0 \end{pmatrix}.$$

Permuting bases, we may also write

$$\tilde{T}_{q,CW}(C^*) = \begin{pmatrix} x_0 & & & & & \\ x_1 & x_0 & 0 & \cdots & & 0 \\ x_2 & 0 & x_0 & & & \\ \vdots & \vdots & & \ddots & & \\ x_q & 0 & \cdots & 0 & x_0 & \\ x_{q+1} & x_1 & \cdots & & x_q & x_0 \end{pmatrix}.$$

Proposition 7.1. $\mathbf{R}(T_{q,CW}) = 2q+1$, $\mathbf{R}(\tilde{T}_{q,CW}) = 2q+3$.

Proof. We first prove the lower bound for $T_{q,CW}$. Apply Proposition 3.1 to show that the rank of the tensor is at least $2q-2$ plus the rank of

$$\begin{pmatrix} 0 & x_0 \\ x_0 & x_1 \end{pmatrix},$$

which has rank 3. An analogous estimate provides the lower bound for $\mathbf{R}(\tilde{T}_{q,CW})$. To show that $\mathbf{R}(T_{q,CW}) \le 2q+1$ consider the following rank 1 matrices, whose span contains $T(A^*)$:

1) $q+1$ matrices with all entries equal to 0 apart from one entry on the diagonal equal to 1,

2) q matrices indexed by $1 \le j \le q$, with all entries equal to zero apart from the four entries $(0,0), (0,j), (j,0), (j,j)$ equal to 1.

For the tensor $\tilde{T}_{CW}$ we consider the same matrices, however both groups have one more element. □

Coppersmith and Winograd used $\tilde{T}_{CW}$ to show $\omega < 2.3755$. In subsequent work Stothers [39], resp. V. Williams [46], resp. LeGall [32] used $\tilde{T}_{CW}^{\otimes 4}$ resp. $\tilde{T}_{CW}^{\otimes 8}$, resp. $\tilde{T}_{CW}^{\otimes 16}$ and $\tilde{T}_{CW}^{\otimes 32}$ leading to the current "world record" $\omega < 2.3728639$.

Ambainis, Filmus and LeGall [2] showed that taking higher powers of $\tilde{T}_{CW}$ when $q \ge 5$ cannot prove $\omega < 2.30$ by this method alone. Their suggestion that one should look for new tensors to prove further upper bounds was one motivation for this paper.

7.3. **Strassen's tensor.** Strassen uses the following concise tensor to show $\omega < 2.48$:

$$T_{Str,q} = \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j \in \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \otimes \mathbb{C}^q \tag{10}$$

which has border rank $q+1$, as the the q vectors $[a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j]$, $1 \le j \le q$, are tangent vectors to q points $[a_0 \otimes b_0 \otimes c_1], \cdots, [a_0 \otimes b_0 \otimes c_q]$ that lie on the $\mathbb{P}^{q-1} = \mathbb{P}\{a_0 \otimes b_0 \otimes \langle c_1, \cdots, c_q \rangle\} \subset Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. Any linear combination of $q+1$ tangent vectors based at $q+1$ linearly dependent points, with any size q subset independent, of any variety, has border rank at most $q+1$, see [31, §10.1]. Here we just take one of the vectors to be zero. Note that $T_{Str,q}$ is a specialization of $T_{CW,q}$ obtained by setting $c_0 = 0$. By the substitution method the rank of the tensor equals $2q$.

The corresponding linear spaces are:

$$T_{Str,q}(C^*) = \begin{pmatrix} 0 & x_1 & \cdots & x_q \\ x_1 & 0 & \cdots & 0 \\ x_2 & & & \\ \vdots & \vdots & & \vdots \\ x_q & 0 & \cdots & 0 \end{pmatrix},$$

and

$$T_{Str,q}(A^*) = \begin{pmatrix} x_1 & x_2 & \cdots & x_q \\ x_0 & 0 & \cdots & 0 \\ 0 & x_0 & 0 & \vdots \\ \vdots & \ddots & & \\ 0 & \cdots & 0 & x_0 \end{pmatrix}.$$

Actually Strassen uses the tensor product of this tensor with its images under $\mathbb{Z}_3$ acting on the three factors: $\tilde{T} := T \otimes T' \otimes T''$ where T', T'' are cyclic permutations of $T = T_{Str,q}$. Thus $\tilde{T} \in \mathbb{C}^{q(q+1)^2} \otimes \mathbb{C}^{q(q+1)^2} \otimes \mathbb{C}^{q(q+1)^2}$ and $\underline{\mathbf{R}}(\tilde{T}) \le (q+1)^3$.

7.4. **Extremal tensors.** Let $A, B, C = \mathbb{C}^m$. There are normal forms for germs of curves in $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ up to order $m-1$, namely

$$T_t = (a_1 + ta_2 + \cdots + t^{m-1}a_m + O(t^m)) \otimes (b_1 + tb_2 + \cdots + t^{m-1}b_m + O(t^m)) \otimes (c_1 + tc_2 + \cdots + t^{m-1}c_m + O(t^m))$$

and if the a_j, b_j, c_j are each linearly independent sets of vectors, we will call the curve *general* to order $m-1$.

Proposition 7.2. *Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$. Let $T_0^{(m-1)}(A^*) := \frac{d^{m-1}}{(dt)^{m-1}}|_{t=0} T_t(A^*)$. If*

$$T(A^*) = T_0^{(m-1)}(A^*),$$

with T_t a curve that is general to order m, then $T(A^*)$ is the centralizer of a regular nilpotent element.

Proof. Note that $T_0^{(q)} = q! \sum_{i+j+k=q-3} a_i \otimes b_j \otimes c_k$, i.e.,

$$T_0^{(q)}(A^*) = \begin{pmatrix} x_{q-2} & x_{q-3} & \cdots & \cdots & x_1 & 0 & \cdots \\ x_{q-3} & x_{q-4} & \cdots & x_1 & 0 & \cdots & \cdots \\ \vdots & & & & & & \\ x_1 & 0 & \cdots & & & & \\ 0 & 0 & \cdots & & & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & & & & \end{pmatrix}$$

in particular, each space contains the previous ones, and the last equals

$$\begin{pmatrix} x_m & x_{m-1} & \cdots & & x_1 \\ x_{m-1} & x_{m-2} & \cdots & x_1 & 0 \\ \vdots & \vdots & \iddots & & \\ \vdots & x_1 & & & \\ x_1 & 0 & & & \end{pmatrix}$$

which is isomorphic to the centralizer of a regular nilpotent element. □

This provides another, explicit proof that the centralizer of a regular nilpotent element belongs to the closure of diagonalizable algebras.

Proposition 7.3. *Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be of border rank $m > 2$. Assume $\mathbb{P}T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) = [X]$ is a single point, and $\mathbb{P}\hat{T}_{[X]} Seg(\mathbb{P}B \times \mathbb{P}C) \supset \mathbb{P}T(A^*)$. Then T is not 1_A-generic.*

Proof. No element of $\mathbb{P}\hat{T}_{[X]} Seg(\mathbb{P}B \times \mathbb{P}C)$ has rank greater than two. □

The purpose of stating Proposition 7.3 is to motivate the following theorem:

Theorem 7.4. *If $\mathbb{P}T(A^*) \cap Seg(\mathbb{P}B \times \mathbb{P}C) = [X]$ is a single point, and $\mathbb{P}\hat{T}_{[X]}Seg(\mathbb{P}B \times \mathbb{P}C) \cap \mathbb{P}T(A^*)$ is a $\mathbb{P}^{m-2}$ and T is 1_A-generic, then $T = \tilde{T}_{m-2,CW}$ is isomorphic to the Coppersmith-Winograd tensor.*

Proof. For the second, we first show that T is 1-generic. If we choose bases such that $X = b_1 \otimes c_1$, then, after changing bases, the $\mathbb{P}^{m-2}$ must be the projectivization of

$$E := \begin{pmatrix} x_1 & x_2 & \cdots & x_{m-1} & 0 \\ x_2 & & & & \\ \vdots & & & & \\ x_{m-1} & & & & \\ 0 & & & & \end{pmatrix}. \tag{11}$$

Write $T(A^*) = \operatorname{span}\{E, M\}$ for some matrix M. As T is 1_A-generic we can assume that M is invertible. In particular, the last row of M must contain a nonzero entry. In the basis order $x_1, \ldots, x_{m-1}, M$, the space of matrices $T(B^*)$ has triangular form and contains matrices with nonzero diagonal entries. The proof for $T(C^*)$ is analogous, hence T is 1-generic.

By Proposition 5.8 we may assume that $T(A^*)$ is contained in the space of symmetric matrices. Hence, we may assume that E is as above and M is a symmetric matrix. By further changing the basis we may assume that M has:

(1) the first row and column equal to zero, apart from their last entries that are nonzero (we may assume they are equal to 1),
(2) the last row and column equal to zero apart from their first entries.

Hence the matrix M is determined by a submatrix M' of rows and columns 2 to $m-1$. As $T(A^*)$ contains a matrix of maximal rank, the matrix M' must have rank $m-2$. We can change the basis $x_2, \ldots, x_{m-1}$ in such a way that the quadric corresponding to M' equals $x_2^2 + \cdots + x_{m-1}^2$. This will also change the other matrices, which correspond to quadrics $x_1 x_i$ for $1 \le i \le m-1$, but will not change the space that they span. We obtain the tensor $\tilde{T}_{m-2,CW}$, that indeed satisfies the assumptions of the theorem. □

7.5. **A second geometric characterization of the Coppersmith-Winograd tensors.** *Compression genericity* is defined and discussed in [30]. Here we just discuss the simplest case. We say a 1-generic, tensor $T \in A \otimes B \otimes C$ is *maximally compressible* if there exists hyperplanes $H_A \subset A^*$, $H_B \subset B^*$, $H_C \subset C^*$ such that $T|_{H_A \times H_B \times H_C} = 0$.

If $T \in S^3 A \subset A \otimes A \otimes A$, we will say T is *maximally symmetric compressible* if there exists a hyperplane $H_A \subset A^*$ such that $T|_{H_A \times H_A \times H_A} = 0$.

Recall that a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ that is 1-generic and satisfies Strassen's equations is strictly isomorphic to a tensor in $S^3 \mathbb{C}^m$.

Theorem 7.5. *Let $T \in S^3 \mathbb{C}^m$ be 1-generic and maximally symmetric compressible. Then T is one of:*

(1) $T_{m-1,CW}$
(2) $\tilde{T}_{m-2,CW}$
(3) $T = a_1(a_1^2 + \cdots a_m^2)$. *As a subspace of $\mathbb{C}^m \otimes \mathbb{C}^m$, this is*

$$\begin{pmatrix} x_1 & x_2 & & \cdots & x_m \\ x_2 & x_1 & 0 & \cdots & \\ x_3 & 0 & x_1 & & \\ \vdots & 0 & & \ddots & \\ x_m & 0 & & & x_1 \end{pmatrix}.$$

In particular, the only 1*-generic, maximally symmetric compressible, minimal border rank tensor in* $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ *is* $\tilde{T}_{m-2,CW}$.

Proof. Let a_1 be a basis of the line ${H_A}^{\perp} \subset \mathbb{C}^m$. Then $T = a_1 Q$ for some $Q \in S^2\mathbb{C}^m$. By 1-genericity, the rank of Q is either m or $m-1$. If the rank is m, there are two cases, either the hyperplane H_A is tangent to Q, or it intersects it transversely. The second is case 3. The first has a normal form $a_1(a_1 a_m + a_2^2 + \cdots + a_{m-1}^2)$, which, when written as a tensor, is $\tilde{T}_{m-2,CW}$. If Q has rank $m-1$, by 1-genericity, its vertex must be in H_A and thus we may choose coordinates such that $Q = (a_2^2 + \cdots + a_m^2)$, but then T, written as a tensor is $T_{m-1,CW}$. □

References

1. Boris Alexeev, Michael A. Forbes, and Jacob Tsimerman, *Tensor rank: some lower and upper bounds*, 26th Annual IEEE Conference on Computational Complexity, IEEE Computer Soc., Los Alamitos, CA, 2011, pp. 283–291. MR 3025382
2. Andris Ambainis, Yuval Filmus, and François Le Gall, *Fast matrix multiplication: Limitations of the laser method*, CoRR **abs/1411.5414** (2014).
3. S. Aronhold, *Theorie der homogenen Functionen dritten Grades von drei Veränderlichen*, J. Reine Angew. Math. **55** (1858), 97–191. MR 1579064
4. Edoardo Ballico and Alessandra Bernardi, *Stratification of the fourth secant variety of Veronese varieties via the symmetric rank*, Adv. Pure Appl. Math. **4** (2013), no. 2, 215–250. MR 3069955
5. Daniel J. Bates and Luke Oeding, *Toward a salmon conjecture*, Exp. Math. **20** (2011), no. 3, 358–370. MR 2836258 (2012i:14056)
6. Markus Bläser, *Explicit tensors*, Perspectives in Computational Complexity, Springer, 2014, pp. 117–130.
7. Grigoriy Blekherman and Zach Teitler, *On maximum, typical and generic ranks*, Math. Ann. **362** (2015), no. 3-4, 1021–1031. MR 3368091
8. Nader H. Bshouty, *On the direct sum conjecture in the straight line model*, J. Complexity **14** (1998), no. 1, 49–62. MR 1617757 (99c:13056)
9. Jaroslaw Buczyński, Adam Ginensky, and J. M. Landsberg, *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*, J. Lond. Math. Soc. (2) **88** (2013), no. 1, 1–24. MR 3092255
10. Jaroslaw Buczyński and J. M. Landsberg, *Ranks of tensors and a generalization of secant varieties*, Linear Algebra Appl. **438** (2013), no. 2, 668–689. MR 2996361
11. ———, *On the third secant variety*, J. Algebraic Combin. **40** (2014), no. 2, 475–502. MR 3239293
12. Jaroslaw Buczynski and Elisa Postinghel, *On strassen's conjecture (lecture)*, https://simons.berkeley.edu/talks/elisa-postinghel-2014-11-12 (2014).
13. Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997, With the collaboration of Thomas Lickteig. MR 99c:68002
14. Enrico Carlini, Maria Virginia Catalisano, and Luca Chiantini, *Progress on the symmetric Strassen conjecture*, J. Pure Appl. Algebra **219** (2015), no. 8, 3149–3157. MR 3320211
15. P. Comon, *Independent Component Analysis, a new concept ?*, Signal Processing, Elsevier **36** (1994), no. 3, 287–314, Special issue on Higher-Order Statistics.
16. Don Coppersmith and Shmuel Winograd, *Matrix multiplication via arithmetic progressions*, J. Symbolic Comput. **9** (1990), no. 3, 251–280. MR 91i:68058
17. Ephraim Feig and Shmuel Winograd, *On the direct sum conjecture*, Linear Algebra Appl. **63** (1984), 193–219. MR 766508 (86h:15022)
18. Ismor Fischer, *Sums of Like Powers of Multivariate Linear Forms*, Math. Mag. **67** (1994), no. 1, 59–61. MR 1573008
19. Shmuel Friedland, *On tensors of border rank l in* $\mathbb{C}^{m\times n\times l}$, Linear Algebra Appl. **438** (2013), no. 2, 713–737. MR 2996364
20. Shmuel Friedland and Elizabeth Gross, *A proof of the set-theoretic version of the salmon conjecture*, J. Algebra **356** (2012), 374–379. MR 2891138
21. Murray Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. of Math. (2) **73** (1961), 324–348. MR 0132079 (24 #A1926)
22. Robert M. Guralnick, *A note on commuting pairs of matrices*, Linear and Multilinear Algebra **31** (1992), no. 1-4, 71–75. MR 1199042 (94c:15021)

23. Wolfgang Hackbusch, *Tensor spaces and numerical tensor calculus*, Springer Series in Computational Mathematics, vol. 42, Springer, Heidelberg, 2012. MR 3236394
24. Atanas Iliev and Laurent Manivel, *Varieties of reductions for* $\mathfrak{gl}_n$, Projective varieties with unexpected properties, Walter de Gruyter GmbH & Co. KG, Berlin, 2005, pp. 287–316. MR MR2202260 (2006j:14056)
25. Joseph Ja'Ja' and Jean Takche, *On the validity of the direct sum conjecture*, SIAM J. Comput. **15** (1986), no. 4, 1004–1020. MR MR861366 (88b:68084)
26. Thomas J. Laffey, *The minimal dimension of maximal commutative subalgebras of full matrix algebras*, Linear Algebra Appl. **71** (1985), 199–212. MR 813045 (87a:15025)
27. J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR 2865915
28. ______, *New lower bounds for the rank of matrix multiplication*, SIAM J. Comput. **43** (2014), no. 1, 144–149. MR 3162411
29. J. M. Landsberg and Laurent Manivel, *Generalizations of Strassen's equations for secant varieties of Segre varieties*, Comm. Algebra **36** (2008), no. 2, 405–422. MR MR2387532
30. J. M. Landsberg and M. Michalek, *A* $2n^2-log(n)-1$ *lower bound for the border rank of matrix multiplication*, ArXiv e-prints (2016).
31. J. M. Landsberg and Zach Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), no. 3, 339–366. MR 2628829 (2011d:14095)
32. Francois Le Gall, *Powers of tensors and fast matrix multiplication*, arXiv:1401.7714.
33. Arielle Leitner, *Limits under conjugacy of the diagonal subgroup in* $SL_n(\mathbb{R})$, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3243–3254. MR 3503693
34. Thomas Lickteig, *Typical tensorial rank*, Linear Algebra Appl. **69** (1985), 95–120. MR 87f:15017
35. Giorgio Ottaviani, *Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited*, Vector Bundles and Low Codimensional Subvarieties: State of the Art and Recent Developments (R. Notari G. Casnati, F. Catanese, ed.), Quaderni di Matematica, vol. 21, Dip. di Mat., II Univ. Napoli, 2007, pp. 315–352.
36. V. Ja. Pan, *On means of calculating values of polynomials*, Uspehi Mat. Nauk **21** (1966), no. 1 (127), 103–134. MR 0207178
37. Ran Raz, *Tensor-rank and lower bounds for arithmetic formulas*, J. ACM **60** (2013), no. 6, Art. 40, 15. MR 3144910
38. A. Schönhage, *Partial and total matrix multiplication*, SIAM J. Comput. **10** (1981), no. 3, 434–455. MR MR623057 (82h:68070)
39. A. Stothers, *On the complexity of matrix multiplication*, PhD thesis, University of Edinburgh, 2010.
40. V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52/53** (1983), 645–685. MR 85b:15039
41. ______, *Relative bilinear complexity and matrix multiplication*, J. Reine Angew. Math. **375/376** (1987), 406–443. MR MR882307 (88h:11026)
42. Volker Strassen, *Evaluation of rational functions*, Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), Plenum, New York, 1972, pp. 1–10, 187–212. MR 0395328
43. ______, *Vermeidung von Divisionen*, J. Reine Angew. Math. **264** (1973), 184–202. MR MR0521168 (58 #25128)
44. D. A. Suprunenko and R. I. Tyshkevich, *Perestanovochnye matritsy*, second ed., Èditorial URSS, Moscow, English translation of first edition: Academic Press: New York, 1968, 2003. MR 2118458 (2006b:16045)
45. Emil Toeplitz, *Ueber ein Flächennetz zweiter Ordnung*, Math. Ann. **11** (1877), no. 3, 434–463. MR 1509924
46. Virginia Williams, *Breaking the coppersimith-winograd barrier*, preprint.
47. Shmuel Winograd, *Some remarks on fast multiplication of polynomials*, Complexity of sequential and parallel numerical algorithms (Proc. Sympos., Carnegie-Mellon Univ., Pittsburgh, Pa., 1973), Academic Press, New York, 1973, pp. 181–196. MR 0375839
48. J. Zuiddam, *A note on the gap between rank and border rank*, ArXiv e-prints (2015).

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