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On the Geometry of Border Rank Decompositions for Matrix Multiplication and Other Tensors with Symmetry*

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Abstract. We present a new approach to study tensors with symmetry, via local algebraic geometry. Border rank decompositions for such tensors—in particular, matrix multiplication and the determinant polynomial—come in families. We prove that these families include representatives with normal forms. These normal forms will be useful to prove lower complexity bounds and possibly even to determine new decompositions. We derive a border rank version of the substitution method used in proving lower bounds for tensor rank. Applying these methods, we improve the lower bound on the border rank of matrix multiplication. We also point out difficulties that will be formidable obstacles to future progress on lower complexity bounds for tensors because of the "wild" structure of the Hilbert scheme of points.

Key words. matrix multiplication complexity, border rank, tensor, commuting matrices, Strassen's equations

AMS subject classifications. 15A69, 14Q20, 14M17

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1. Introduction.

Ever since Strassen discovered in 1969 [35] that the standard algorithm for multiplying matrices is not optimal, it has been a central question to determine upper and lower bounds for the complexity of the matrix multiplication tensor $M_{\langle \mathbf{n} \rangle} \in \mathbb{C}^{\mathbf{n}^2} \otimes \mathbb{C}^{\mathbf{n}^2} \otimes \mathbb{C}^{\mathbf{n}^2}$. Using algebraic geometry, Strassen [36] proved the first nonclassical complexity lower bound for $M_{\langle \mathbf{n} \rangle}$. After that, other than an improvement in the error term by Lickteig [30], there was a hiatus of 30 years for general $\mathbf{n}$, until [27], where further use of algebraic geometry led to new lower complexity bounds. In geometric language, the complexity of matrix multiplication is governed by the smallest value r such that the matrix multiplication tensor lies on the rth secant variety of the Segre variety $Seg(\mathbb{P}^{\mathbf{n}^2-1} \times \mathbb{P}^{\mathbf{n}^2-1} \times \mathbb{P}^{\mathbf{n}^2-1})$—see below for definitions. This value of r is called the *border rank* of $M_{\langle \mathbf{n} \rangle}$ and is denoted $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle})$. All general results on the complexity of matrix multiplication since 1970, both upper and lower complexity bounds, have been obtained using border rank. Thus, the border rank is the standard complexity measure for matrix multiplication.

In this paper we utilize symmetry groups to go from algebraic geometry to local algebraic geometry. We show that for matrix multiplication and other tensors with symmetry, instead of determining membership in secant varieties, one can determine membership in local variants,

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which we call *greater areoles*. Our main results, Lemma 3.1, Proposition 3.2, and Corollary 4.4, prove that these varieties also govern the complexity of matrix multiplication and they have the advantage of being proper subvarieties of the secant variety by Proposition 4.13. The upshot is that border rank decompositions for matrix multiplication can be reduced to decompositions of a very special form and we develop basic language to study them.

After establishing a border rank analogue of the substitution method in section 5, we illustrate the utility of this approach by using the group action to improve the known lower bounds for $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle})$ by one. In later work, after this paper was posted on arXiv in [26], we made further use of this approach to improve the lower bound to $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - \log_2(\mathbf{n}) - 1$. Strassen's 1983 bound [36] was $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq \frac{3}{2}\mathbf{n}^2$, Lickteig's 1985 bound [30] was $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq \frac{3}{2}\mathbf{n}^2 + \frac{\mathbf{n}}{2} - 1$, the 2011 bound of [27] was $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - \mathbf{n}$, and in this paper we show $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - \mathbf{n} + 1$.

From the perspective of algebraic geometry, our restriction amounts to reducing the study of the Hilbert scheme of points to the punctual Hilbert scheme (those schemes supported at a single point).

While motivated by the complexity of matrix multiplication, our work fits into both the larger study of the structure of secant varieties of homogeneous varieties (e.g., [37, 9]) and the study of the geometry of tensors (e.g., [22]).

Overview. In section 2 we define secant varieties and varieties of border rank decompositions. In section 3 we prove our main normal form lemma and show that it applies to the problems of studying the Waring border rank of the determinant and the tensor border rank of the matrix multiplication operator. To better study the normal forms, in section 4 we define subvarieties of secant varieties corresponding to these normal forms—further applications of such constructions were found in [31, 8]. In section 5 we prove a border rank version of the substitution method as used in [1]. As an application we show that $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - \mathbf{n} + 1$, an improvement by one over the previous lower bound of [27].

Notation. Throughout this paper, $\mathbf{V}, A, B, C, U, V, W$ denote complex vector spaces and $X \subset \mathbb{P}\mathbf{V}$ denotes a projective variety. If $v \in \mathbf{V}$, we let $[v] \in \mathbb{P}\mathbf{V}$ denote the corresponding point in projective space. For a variety X, $X^{(r)} = X^{\times r}/\mathfrak{S}_r$ denotes the variety of r-tuples of points of X, where $\mathfrak{S}_r$ is the group of permutations on r elements. The vector space of linear maps $U \to V$ is denoted $U^*{\otimes}V$. If $X \subset \mathbb{P}\mathbf{V}$ is a set, define the stabilizer of X, $G_X := \{g \in GL(\mathbf{V}) \mid g \cdot X = X\}$. We also write $G_{X,Y} = G_X \cap G_Y$.

2. Secant varieties.

For a variety $X \subset \mathbb{P}\mathbf{V}$, let

$$\sigma_r^0(X) = \bigcup_{x_1,\dots,x_r \in X} \langle x_1, \dots, x_r \rangle \subset \mathbb{P}\mathbf{V}$$

denote the points of $\mathbb{P}\mathbf{V}$ on secant $\mathbb{P}^{r-1}$'s of X, and let $\sigma_r(X) := \overline{\sigma_r^0(X)}$ denote its Zariski closure, the *rth secant variety* of X, where $\langle x_1, \dots, x_r \rangle$ denotes the projective linear space spanned by the points $x_1, \dots, x_r$ (usually it is a $\mathbb{P}^{r-1}$).

In this paper we are primarily concerned with the case $\mathbb{P}\mathbf{V} = \mathbb{P}(A{\otimes}B{\otimes}C)$ and $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is the *Segre variety* of rank one tensors. The above-mentioned question about the matrix multiplication tensor is the case $A = U^*{\otimes}V$, $B = V^*{\otimes}W$, and $C = W^*{\otimes}U$,

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and $M_{\langle U,V,W\rangle} \in A\otimes B\otimes C$ is the matrix multiplication tensor. Consider the identity maps $\mathrm{Id}_U \in U^*\otimes U$, $\mathrm{Id}_V \in V^*\otimes V$, $\mathrm{Id}_W \in W^*\otimes W$; then $M_{\langle U,V,W\rangle} = \mathrm{Id}_U \otimes \mathrm{Id}_V \otimes \mathrm{Id}_W$. When $U, V, W = \mathbb{C}^n$, we denote $M_{\langle U,V,W\rangle}$ by $M_{\langle \mathbf{n}\rangle}$. The question is: What is the smallest r such that $[M_{\langle U,V,W\rangle}] \in \sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$? Bini [3] showed that this r, called the *border rank* of $M_{\langle U,V,W\rangle}$, indeed governs its complexity. The border rank of a tensor T is denoted $\underline{\mathbf{R}}(T)$. The smallest r such that a tensor $[T] \in \mathbb{P}(A\otimes B\otimes C)$ is in $\sigma_r^0(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$ is called the *rank* of T and is denoted $\mathbf{R}(T)$.

Remark 2.1. It is expected that the rank of $M_{\langle \mathbf{n}\rangle}$ is greater than its border rank when $\mathbf{n} > 2$. However, all we know is that $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2 - O(\mathbf{n})$ and $\mathbf{R}(M_{\langle \mathbf{n}\rangle}) \geq 3\mathbf{n}^2 - o(\mathbf{n}^2)$ [27, 23], and when $\mathbf{n} = 3$, $19 \leq \mathbf{R}(M_{\langle 3\rangle}) \leq 23$ while $16 \leq \underline{\mathbf{R}}(M_{\langle 3\rangle}) \leq 20$ with inequalities proved, respectively, in [4, 20], this paper, and [33].

Definition 2.2. *Let $X \subset \mathbb{P}\mathbf{V}$ be a projective variety. By an X-*border rank r decomposition *for $z \in \mathbb{P}\mathbf{V}$, we mean a curve E_t in the Grassmannian $G(r, \mathbf{V})$ such that $z \in \mathbb{P}E_0$, and for $t > 0$, E_t is spanned by r points of X. (This includes the possibility of E_t being stationary.) In particular, z admits an X-border rank r decomposition if and only if $z \in \sigma_r(X)$. When $X = Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, we just refer to* border rank decompositions. *We will say such an* E_0 realizes *z as a point of $\sigma_r(X)$.*

Remark 2.3. Instead of taking a curve in the Grassmannian we may, and sometimes will, just take a convergent sequence E_{t_n}.

Define the *incidence variety*

$$S_r^0(X) := \{([v], ([x_1], \dots, [x_r])) \mid v \in \langle x_1, \dots, x_r\rangle\} \subset \mathbb{P}\mathbf{V} \times X^{(r)},$$

a "Nash"-type blow-up of it

$$\tilde{S}_r^0(X) := \{([v], ([x_1], \dots, [x_r]), \langle x_1, \dots, x_r\rangle) \mid v \in \langle x_1, \dots, x_r\rangle, \dim\langle x_1, \dots, x_r\rangle = r\}$$
$$\subset \mathbb{P}\mathbf{V} \times X^{(r)} \times G(r, \mathbf{V}),$$

and the (blown up) *abstract secant variety*

$$S_r(X) := \overline{\tilde{S}_r^0(X)}.$$

We have maps

$$\begin{array}{ccc} & S_r(X) & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ G(r, \mathbf{V}) & & \sigma_r(X), \end{array}$$

where the map π is surjective.

When discussing rank decompositions, a point $p \in \sigma_r^0(X)$ is called *identifiable* if there is a unique collection of r points of X such that p is in their span. When discussing border rank realizations, the r-plane E_0 is the more important object, which motivates the following definition.

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Definition 2.4. *We say* $[v] \in \sigma_r(X)$ *is* Grassmann-border-identifiable *if* $\rho\pi^{-1}([v])$ *is a point.*

We will be mostly interested in the case when $X = G/P \subset \mathbb{P}\mathbf{V}$ is a homogeneous variety and $[v]$ has a nontrivial symmetry group $G_v \subset G = G_X$. In this case $[v]$ is almost never Grassmann-border-identifiable. Indeed, if $z \in \pi^{-1}([v])$, then the orbit closure $\overline{G_v \cdot z}$ is also in $\pi^{-1}([v])$. Hence, to be Grassmann-border-identifiable, G_v would have to act trivially on $\rho(z)$.

3. The normal form lemma. By [9, Lemma 2.1], in any border rank decomposition with $X = G/P \subset \mathbb{P}\mathbf{V}$ homogeneous, we may assume there is one stationary point $x \in X$ with $x \in \mathbb{P}E_t$ for all t.

The following lemma is central.

Lemma 3.1 (normal form lemma). *Let* $X = G/P \subset \mathbb{P}\mathbf{V}$*, and let* $v \in \mathbf{V}$ *be such that* G_v *has a single closed orbit* $\mathcal{O}_{min}$ *in* X*. Then any border rank* r *decomposition of* v *may be modified to a border rank* r *decomposition* $E = \lim_{t\to 0}\langle x_1(t), \dots, x_r(t)\rangle$ *where there is a stationary point* $x_1(t) \equiv x_1$ *lying in* $\mathcal{O}_{min}$*.*

If, moreover, every orbit of G_{v,x_1} *contains* x_1 *in its closure, we may further assume that all other* $x_j(t)$ *limit to* x_1*.*

Proof. The proof of the first statement follows from the same methods as the proof of the second; hence we focus on the latter. We prove we can have all points limiting to the same point $x_1(0)$. By [9, Lemma 2.1] this is enough to conclude.

We work by induction. Say we have shown that $x_1(t), \dots, x_q(t)$ all limit to the same point $x_1 \in \mathcal{O}_{min}$. We will show that our curve can be modified so that the same holds for $x_1(t), \dots, x_{q+1}(t)$. Take a curve $g_\epsilon \in G_{v,x_1}$ such that $\lim_{\epsilon\to 0} g_\epsilon x_{q+1}(0) = x_1$. For each fixed ϵ, acting on the $x_j(t)$ by g_ϵ, we obtain a border rank decomposition for which $g_\epsilon x_i(t) \to g_\epsilon x_1(0) = x_1(0)$ for $i \le q$ and $g_\epsilon x_{q+1}(t) \to g_\epsilon x_{q+1}(0)$. Fix a sequence $\epsilon_n \to 0$. Claim: We may choose a sequence $t_n \to 0$ such that

- $\lim_{n\to\infty} g_{\epsilon_n} x_{q+1}(t_n) = x_1(0)$,
- $\lim_{n\to\infty} < g_{\epsilon_n} x_1(t_n), \dots, g_{\epsilon_n} x_r(t_n) >$ contains v, and
- $\lim_{n\to\infty} g_{\epsilon_n} x_j(t_n) = x_1(0)$ for $j \le q$.

The first point holds as $\lim_{\epsilon\to 0} g_\epsilon x_{q+1}(0) = x_1$. The second follows as for each fixed ϵ_n, taking t_n sufficiently small we may ensure that a ball of radius $1/n$ centered at v intersects $\langle g_{\epsilon_n} x_1(t_n), \dots, g_{\epsilon_n} x_r(t_n)\rangle$. In the same way we may ensure that the third point is satisfied. Considering the sequence $\tilde{x}_i(t_n) := g_{\epsilon_n} x_i(t_n)$, we obtain the desired border rank decomposition. ∎

We are mainly interested in the following two examples.

3.1. The determinant polynomial. Let $v_n : \mathbb{P}W \to \mathbb{P}(S^n W)$ denote the Veronese embedding of $\mathbb{P}W$. When $W = E \otimes F = \mathbb{C}^n \otimes \mathbb{C}^n$, the space $\mathbf{V} = S^n W$ is the home of the determinant polynomial. Write $X = v_n(\mathbb{P}W) \subset \mathbb{P}S^n W$ and $v = \det_n$ for the determinant. Here $G_X = GL_{n^2}$ and $G_{\det_n} \simeq (SL(E) \times SL(F)) \ltimes \mathbb{Z}_2$. The group $G_{\det_n}$ has a unique closed orbit $\mathcal{O}_{min} = v_n(Seg(\mathbb{P}E \times \mathbb{P}F))$ in X. Moreover, for any $z \in v_n(Seg(\mathbb{P}E \times \mathbb{P}F))$, $G_{\det_n, z}$, the group preserving both $\det_n$ and z, is isomorphic to $P_E \times P_F$, where P_E, P_F are the parabolic subgroups of matrices with zero in the first column except the $(1,1)$-slot, and z is in the

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$G_{\det_n,z}$-orbit closure of any $q \in v_n(\mathbb{P}W)$.

3.2. The matrix multiplication tensor. Set $A = U^*{\otimes}V$, $B = V^*{\otimes}W$, and $C = W^*{\otimes}U$. The space $\mathbf{V} = A{\otimes}B{\otimes}C$ is the home of the matrix multiplication tensor, $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) = Seg(\mathbb{P}(U^*{\otimes}V) \times \mathbb{P}(V^*{\otimes}W) \times \mathbb{P}(W^*{\otimes}U)) \subset \mathbb{P}(A{\otimes}B{\otimes}C)$, and $v = M_{\langle U,V,W\rangle} = \mathrm{Id}_U \otimes \mathrm{Id}_V \otimes \mathrm{Id}_W \in (U^*{\otimes}V){\otimes}(V^*{\otimes}W){\otimes}(W^*{\otimes}U)$ is the matrix multiplication tensor. Here $G_X = GL(A) \times GL(B) \times GL(C) \subset GL(\mathbf{V})$ if the spaces are of different dimensions and $G_X = GL(A) \times GL(B) \times GL(C) \rtimes \mathfrak{S}_3 \subset GL(\mathbf{V})$ if they all have the same dimension, and $G_{M_{\langle U,V,W\rangle}} = PGL(U) \times PGL(V) \times PGL(W) \subset G_X$ if the spaces all have different dimensions and $G_{M_{\langle U,V,W\rangle}} = PGL(U) \times PGL(V) \times PGL(W) \rtimes (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \subset G_X$ if they all have the same dimension; see, e.g., [12].

Proposition 3.2. *Let*

$$\mathcal{K} := \{[\mu{\otimes}v{\otimes}\nu{\otimes}w{\otimes}\omega{\otimes}u] \in Seg(\mathbb{P}U^* {\times} \mathbb{P}V {\times} \mathbb{P}V^* {\times} \mathbb{P}W {\times} \mathbb{P}W^* {\times} \mathbb{P}U) \mid \mu(u) = \omega(w) = \nu(v) = 0\}.$$

Then $\mathcal{K}$ is the unique closed $G_{M_{\langle U,V,W\rangle}}$-orbit in $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

Moreover, if $k \in \mathcal{K}$, then $G_{M_{\langle U,V,W\rangle},k}$, the group preserving both $M_{\langle U,V,W\rangle}$ and k, is such that for every $p \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, $k \in \overline{G_{M_{\langle U,V,W\rangle},k} \cdot p}$.

Note that $Seg(\mathbb{P}U \times \mathbb{P}U^*)_0 := \{[u{\otimes}\alpha] \mid \alpha(u) = 0\} \subset \mathbb{P}\mathfrak{sl}(U)$ is the closed orbit in the adjoint representation and $\mathcal{K}$ is isomorphic to $Seg(Seg(\mathbb{P}U \times \mathbb{P}U^*)_0 \times Seg(\mathbb{P}V \times \mathbb{P}V^*)_0 \times Seg(\mathbb{P}W \times \mathbb{P}W^*)_0)$.

Proof. It is enough to prove the last statement. We will prove that $\mathcal{K}$ is the unique closed orbit under $G_{M_{\langle U,V,W\rangle},k}$. This is enough to conclude as the closure of any orbit must contain a closed orbit. Notice that fixing $k = [(\mu{\otimes}v){\otimes}(\nu{\otimes}w){\otimes}(\omega{\otimes}u)]$ is equivalent to fixing a partial flag in each U, V, and W consisting of a line and a hyperplane containing it.

Let $[a{\otimes}b{\otimes}c] \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$. If $[a] \notin Seg(\mathbb{P}U^* \times \mathbb{P}V)$, then the orbit is not closed, even under the torus action on V or U^* that is compatible with the flag. So without loss of generality, we may assume $[a{\otimes}b{\otimes}c] \in Seg(\mathbb{P}U^* \times \mathbb{P}V \times \mathbb{P}V^* \times \mathbb{P}W \times \mathbb{P}W^* \times \mathbb{P}U)$. Write $a{\otimes}b{\otimes}c = (\mu'{\otimes}v'){\otimes}(\nu'{\otimes}w'){\otimes}(\omega'{\otimes}u')$. If, for example, $v' \neq v$, we may act with an element of $GL(V)$ that preserves the partial flag and sends v' to $v + \epsilon v'$. Hence v is in the closure of the orbit of v'. As $G_{M_{\langle U,V,W\rangle},k}$ preserves v we may continue, reaching k in the closure. ∎

Remark 3.3. Proposition 3.2 combined with the normal form lemma allows the argument of [21] to be simplified tremendously. The original proof involved treating sixteen cases. Our result immediately enables one to eliminate thirteen of them. In particular, it eliminates the need for the erratum.

4. Local versions of secant varieties. In this section we introduce several higher order generalizations of the tangent star at a point of a variety and the tangential variety. These generalizations are subvarieties of the rth secant variety of a projective variety. We restrict our discussion to projective varieties $X \subset \mathbb{P}\mathbf{V}$; however, the definitions can be extended to arbitrary embedded schemes. Our initial motivation was to provide language to discuss the normal form of Lemma 3.1, but the discussion is useful in a wider context; special cases have already been used in [8, 31].

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We exhibit local properties of the rth secant variety using the language of smoothable schemes of length r supported at one point (i.e., local). (The length of a zero dimensional scheme R is $\dim_{\mathbb{C}} \mathcal{O}_R(R)$.) Their moduli space is in the principal component of the Hilbert scheme of subschemes of length r of X. This component is an algebraic variety, a compactification of r-tuples of distinct points of X, i.e., $(X^{(r)} \setminus D)$, where D is the big diagonal. It parameterizes smoothable schemes, i.e., schemes that arise as degenerations of r distinct points with reduced structure. More precisely, an ideal I defines a smoothable scheme if there exists a flat family I_t with the fiber I for $t = 0$, where for $t \neq 0$, I_t is the ideal of r distinct points.

4.1. Areoles and buds. Recall that a scheme S supported at $0 \in \mathbb{C}^n$ corresponds to an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ whose only zero is (0). Define the span of S to be $\langle S \rangle := \mathrm{Zeros}(I_1)$, where $I_1 \subset I$ is the homogeneous degree one component. This definition depends on the embedding of S.

We start by recalling the definition of the areole from [8, section 5.1].

Definition 4.1 (areole). *Let $p \in X \subset \mathbb{P}\mathbf{V}$. The* rth open areole *at p is*

$$\mathfrak{a}_r^\circ(X,p) := \bigcup \{\langle R \rangle \mid R \textit{ is smoothable in } X, \textit{ supported at } p \textit{ and } \mathrm{length}(R) \leq r\} \subset \mathbb{P}\mathbf{V},$$

the rth areole *at p is the closure*

$$\mathfrak{a}_r(X,p) := \overline{\mathfrak{a}_r^\circ(X,p)} \subset \mathbb{P}\mathbf{V},$$

and the rth areole variety *of X is*

$$\mathfrak{a}_r(X) := \overline{\bigcup_{p \in X} \mathfrak{a}_r(X,p)} \subset \mathbb{P}\mathbf{V}.$$

The areole can be regarded as a generalization of a tangent space. Indeed, consider $r = 2$ and a smooth point $p \in X$. Up to isomorphism there is only one local scheme of length two: $\mathrm{Spec}\, \mathbb{C}[x]/(x^2)$, and the embedded tangent space at p may be identified with linear spans of such schemes, supported at p. In particular, $\mathfrak{a}_2(X) = \tau(X)$, the tangential variety of X; see Proposition 4.12.

Another, differential geometric, definition of a tangent line is as a limit of secant lines. This motivates the following definition.

Definition 4.2 (greater areole). *The* rth open greater areole *at p is*

$$\tilde{\mathfrak{a}}_r^\circ(X,p) := \bigcup_{\substack{x_j(t) \subset X \\ x_j(t) \to p}} \lim_{t \to 0} \langle x_1(t), \ldots, x_r(t) \rangle \subset \mathbb{P}\mathbf{V},$$

the rth greater areole *at p is the closure*

$$\tilde{\mathfrak{a}}_r(X,p) := \overline{\tilde{\mathfrak{a}}_r^\circ(X,p)} \subset \mathbb{P}\mathbf{V},$$

and the rth greater areole variety *of X is*

$$\tilde{\mathfrak{a}}_r(X) := \overline{\bigcup_{p \in X} \tilde{\mathfrak{a}}_r(X,p)} \subset \mathbb{P}\mathbf{V}.$$

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Remark 4.3. The difference between the areole and the greater areole is related to the difference of border rank and smoothable rank—the latter was introduced in [32]. Indeed, points in the rth open areole belong to a linear span of a smoothable scheme and, hence, are of smoothable rank at most r.

The normal form in Lemma 3.1 can be restated in the following way. If a point v satisfies all assumptions of the lemma, then it belongs to the rth secant variety if and only if it belongs to the rth greater areole $\tilde{\mathfrak{a}}_r(X,p)$ for a point $p \in \mathcal{O}_{min}$.

In particular, we have the following corollary.

Corollary 4.4. *The matrix multiplication tensor $M_{\langle \mathbf{n} \rangle}$ has border rank at most r if and only if $M_{\langle \mathbf{n} \rangle} \in \tilde{\mathfrak{a}}_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C, k)$, where k is any point as in Proposition 3.2.*

Proof. The proof follows by Lemma 3.1 and Proposition 3.2. ∎

Lemma 4.5. *$\mathfrak{a}^\circ_r(X,p) \subset \tilde{\mathfrak{a}}^\circ_r(X,p)$ and $\mathfrak{a}_r(X,p) \subset \tilde{\mathfrak{a}}_r(X,p)$.*

Proof. It is enough to show the first inclusion. Let $v \in \mathfrak{a}^\circ_r(X,p)$. Then, by definition, there exists a scheme S, smoothable in X, supported at p such that $v \in \langle S \rangle$. As S is smoothable in X we may find a family of points $x_i(t)$ for $i = 1,\dots,k$, such that S is their limit as $t \to 0$. We may also assume that $\langle x_1(t),\dots,x_r(t)\rangle$ is of constant dimension $r-1$. Let $E := \lim_{t\to 0}\langle x_1(t),\dots,x_r(t)\rangle$ that is also of dimension $r-1$. The linear span $\langle S \rangle$ of the limit is contained in the limit E of linear spans. By definition E is contained in $\tilde{\mathfrak{a}}^\circ_r(X,p)$. ∎

Remark 4.6. The relation between the linear span of the limit and the limit of linear spans can be viewed as a special case of upper semicontinuity of Betti numbers under deformation [15, III.12.8], i.e., the number of equations of fibers in a flat family of given degree can only jump up in the limit.

The cases when the areole equals the greater areole are of particular interest.

The following proposition is well known; however, it is usually stated in different language. It is essentially due to Grothendieck [14] and played an important role in the construction of the Hilbert scheme. Recently, it was crucial in [8]. The proof of the following proposition follows from [8, Theorem 5.7].

Proposition 4.7. *Suppose that p is a point of a variety X embedded by at least an $(r-1)$st Veronese embedding. Then*

$$\mathfrak{a}_r(X,p) = \tilde{\mathfrak{a}}_r(X,p).$$

Because we are interested in secant varieties, the definition of the areole involves only smoothable local schemes. It might seem that the classification of such local schemes should be easy and that they should all be "almost" like $\operatorname{Spec}\mathbb{C}[x]/(x^r)$. As we discuss below, the story is much more interesting.

In the Hilbert scheme, the locus of schemes supported at p and isomorphic to $\operatorname{Spec}\mathbb{C}[x]/(x^r)$ is an open subset of the punctual Hilbert scheme. Such schemes are called *aligned* [17] or *curvilinear*. The schemes in the closure in the Hilbert scheme of the locus of aligned schemes, i.e., schemes that arise as degenerations of aligned schemes, are called *alignable*. For small values of r all local smoothable schemes are alignable. An example of a scheme that is alignable, but not aligned, is $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,y^2)$.

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A local scheme $\operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]/I$ is called *Gorenstein* if the ideal I is an *apolar ideal* of a polynomial f in the dual variables. That is, the x_i are differential operators on the dual space of polynomials, and I is the ideal of differential operators annihilating f.

The Hilbert function of a local scheme with a maximal ideal $\mathfrak{m}$ assigns to k the dimension of $\mathfrak{m}^k/\mathfrak{m}^{k+1}$. By convention (which we follow), it is usually presented as a finite sequence, with the infinite string of zeros after the last nonzero entry omitted.

The last nonzero value of the Hilbert function for a Gorenstein scheme must be equal to 1, because if the form f is of degree d, the pairing with differential operators of degree d provides a surjection onto $\mathbb{C}$.

Example 4.8. The scheme $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,xy,y^2)$ is not Gorenstein, as its Hilbert function equals $(1,2)$. The scheme $\operatorname{Spec}\mathbb{C}[x,y]/(x^2-y^2,xy)$ is Gorenstein as the ideal is apolar to X^2+Y^2, where X,Y is the dual basis to x,y.

Schemes that are local and smoothable do not have to be alignable [6]. Furthermore, there exist subvarieties of the Hilbert scheme corresponding to local, smoothable schemes that are of higher dimension than the component of alignable schemes. When X is n dimensional the dimension of the locus of alignable schemes equals $(r-1)(n-1)$. Already for $r=12$ and $n=5$, there exists another family (also of dimension 44) of smoothable schemes supported at p that are not alignable. For $r=16$ and $n=7$ there exists a family of dimension 104 of smoothable schemes supported at p [2, 19]. Summing over different p we obtain a family of dimension $111 = 7\cdot 16 - 1$, i.e., a divisor in the Hilbert scheme!

We make one more definition of a subvariety of the areole that we expect to be more tractable than the areole itself.

Definition 4.9 (bud). *The* rth open bud *at* $p \in X \subset \mathbb{P}\mathbf{V}$ *is*

$$\mathfrak{b}_r^\circ(X,p) := \bigcup\{\langle R\rangle \mid R \simeq \operatorname{Spec}\mathbb{C}[x]/(x^r) \text{ and supported at } p\} \subset \mathbb{P}\mathbf{V},$$

the rth bud *at* p *is the closure*

$$\mathfrak{b}_r(X,p) := \overline{\mathfrak{b}_r^\circ(X,p)} \subset \mathbb{P}\mathbf{V},$$

and the rth bud variety *of* X *is*

$$\mathfrak{b}_r(X) := \overline{\bigcup_{p\in X} \mathfrak{b}_r(X,p)} \subset \mathbb{P}\mathbf{V}.$$

Remark 4.10. If X is a homogeneous variety, then the bud, areole, and greater areole varieties do not need closures in their definitions. Indeed, by the group action, in such a case the buds, areoles, and greater areoles at all points are isomorphic. Further, if all smoothable (resp., alignable) local schemes of length r, supported at a point p, have linear spans of maximal dimension $r-1$, then the areole (resp., bud) at p does not need the closure in its definition. This is the case, for example, when the variety is embedded by at least $(r-1)$st Veronese.

Note that the bud $\mathfrak{b}_r(X,p)$ contains the linear spans of all alignable schemes of given length that are supported on p. These schemes do not have to be Gorenstein. However, of course the aligned schemes are Gorenstein.

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Example 4.11. In [25] we showed that when $\dim A = \dim B = \dim C = m$, then $\mathfrak{b}_m(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = \overline{GL(A) \times GL(B) \times GL(C) \cdot [T_N]} \subset \mathbb{P}(A \otimes B \otimes C)$, where T_N is a tensor such that $T_N(A^*) \subset B \otimes C$ corresponds, after a choice of $\alpha \in A^*$ with $\operatorname{rank} T(\alpha) = m$, to the centralizer of a regular nilpotent element.

Both areoles and buds generalize the *tangent star* [37], which is the case $r = 2$.

Proposition 4.12. *Let p be a point of a variety $X \subset \mathbb{P}\mathbf{V}$. Then*

$$\mathfrak{b}_2(X, p) = \mathfrak{a}_2(X, p) = \tilde{\mathfrak{a}}_2(X, p) = T^{\star}_p X,$$

where $T^{\star}_p X$ denotes the tangent star of X at p.

Proof. The first equality follows by definition as any local scheme of length two is isomorphic to $\mathbb{C}[x]/(x^2)$, i.e., aligned. The second equality is a special case of Proposition 4.7 as any variety is its own first Veronese re-embedding. The third is just the definition of the tangent star. ∎

Thus when $r = 2$, $\mathfrak{b}_2(X) = \mathfrak{a}_2(X) = \tilde{\mathfrak{a}}_2(X)$. Moreover, $\mathfrak{b}^0_2(X) = \mathfrak{b}_2(X)$ because all alignable schemes of length two are aligned.

Suppose $r = 3$ and $X = G/P$ is generalized cominuscule—for a precise definition we refer to [9, sect. 3]. Here, we just mention that arbitrary products of Grassmannians under arbitrary homogeneous embeddings (hence also Segre–Veronese varieties) are generalized cominuscule. By [9, Thm. 1.11] the three varieties still coincide: $\mathfrak{b}_3(G/P) = \mathfrak{a}_3(G/P) = \tilde{\mathfrak{a}}_3(G/P)$. Moreover, $\mathfrak{b}^0_3(G/P) = \mathfrak{b}_3(G/P)$ because all the points in the bud give aligned schemes.

We now bound the dimensions of all these varieties. Recall that for an n dimensional variety, $\dim \sigma_r(X) \le rn + r - 1$.

Proposition 4.13. *Let p be a point of an n dimensional homogeneous variety $X \subset \mathbb{P}^N$. Then*

$$\begin{aligned}\dim \tilde{\mathfrak{a}}_r(X, p) &\le rn - n + r - 2,\\ \dim \mathfrak{a}_r(X, p) &\le rn - n + r - 2,\\ \dim \mathfrak{b}_r(X, p) &\le (r-1)n,\end{aligned}$$

and hence,

$$\begin{aligned}\dim \tilde{\mathfrak{a}}_r(X) &\le rn + r - 2,\\ \dim \mathfrak{a}_r(X) &\le rn + r - 2,\\ \dim \mathfrak{b}_r(X) &\le rn.\end{aligned}$$

Proof. The second inequality follows by simply bounding the dimension of the locus of punctual smoothable schemes as a divisor in the Hilbert scheme. The third inequality follows as the locus of alignable schemes is of dimension $(r-1)(n-1)$.

To prove the first inequality consider the projection $pr : S_r(X) \to X^{(r)} \times G(r, N+1)$. The intersection of $pr(S_r(X))$ with the small diagonal in $X^{(r)}$ times the Grassmannian is at most a divisor. Since X is homogeneous, the fibers of the projection of the intersection to the small diagonal are all isomorphic and hence are of dimension at most $nr - n - 1$. The inequality follows. ∎

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Remark 4.14. The only point in the proof where we used that X was homogeneous was to have equidimensional fibers. We expect that the inequalities remain true for any smooth X.

While the areole and greater areole have the same expected dimension, in many cases (for example, $r \leq 9$) the areole is of strictly smaller dimension than expected, often coinciding with the bud. Further, the areole and the greater areole are not expected to be irreducible. The problem of distinguishing between the areole and greater areole appears to be important and difficult. On the other hand, the bud is irreducible, as the locus of aligned schemes is irreducible in the Hilbert scheme.

By the inequalities in Proposition 4.13, $\tilde{\mathfrak{a}}_r(X)$, $\mathfrak{a}_r(X)$, and $\mathfrak{b}_r(X)$ are all proper subvarieties of the secant variety when $\sigma_r(X)$ has the expected dimension. Finding the equations of any of the above varieties when $r > 2$, even in the case of Segre or Veronese varieties, is an important and difficult challenge.

4.2. The bud and local differential geometry. We thank Jaroslaw Buczyński and Joachim Jelisiejew for pointing us towards the following result.

Proposition 4.15. *Let $X \subset \mathbb{P}\mathbf{V}$ be a projective variety, and let $p \in X$ be a smooth point. Then*

$$\mathfrak{b}_r(X,p) = \overline{\bigcup_{\substack{x(t)\subset X\\ x(t)\to p}} \langle x(0), x'(0), \ldots, x^{(r-1)}(0)\rangle},$$

where the union is taken over all curves $x(t)$ smooth at p.

In the language of differential geometry, $\mathfrak{b}_r(X,p)$ is the $(r-1)$*st osculating cone to X at p*. Its span is the $(r-1)$st osculating space. Thus buds provide an algebraic definition of osculating spaces.

Remark 4.16. We obtain the same variety if we take the union over analytic curves.

Proof. Consider a curve C in X that is smooth at p. For each r, there is an embedded aligned subscheme of C (hence, also of X) of length at most r that is supported at p and its linear span equals $\langle x(0), x'(0), \ldots, x^{(r-1)}(0)\rangle$.

Given an aligned scheme S, we claim there exists a curve that contains it, is smooth at p, and is contained in X. Let $\mathfrak{m}$ be the maximal ideal defining p in $\mathcal{O}(X)_p$, the local ring of $p \in X$. Let J be the ideal defining S in $\mathcal{O}(X)_p$. Since the tangent space of S is one dimensional, $(J + \mathfrak{m}^2)/\mathfrak{m}^2$ is a hyperplane in $\mathfrak{m}/\mathfrak{m}^2 = T^*_p X$. Let the equivalence classes of $f_1, \ldots, f_{\dim X-1} \in J$ span this hyperplane. Locally, we may write $f_i = h_i/s_i$ with $h_i \in I(S)$ and $s_i \in \mathbb{C}[\mathbf{V}]$ with $s_i(p) \neq 0$. Let $U = X \setminus \bigcup_i \mathrm{Zeros}(s_i)$. Then the h_i are lifts of the f_i to $\mathbb{C}[U]$. Consider the subscheme (possibly reducible, nonreduced) $Z \subset U$ they define. The Zariski tangent space T_pZ is one dimensional, so Z must also be locally one dimensional at p (at most one dimensional because the local dimension is at most the dimension of T_pZ, and at least because we used $\dim X - 1$ equations); hence a component through p must be a curve, smooth at p, and this curve has the desired properties. ■

For a subvariety $X \subset \mathbb{P}\mathbf{V}$ and a smooth point $x \in X$, there is a sequence of differential invariants called the *fundamental forms* $\mathbb{FF}_k : S^kT_xX \to N^j_xX$, where N^j_xX is the jth normal space. After making choices of splittings and ignoring twists by line bundles, write $\mathbf{V} =$

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$\hat{x} \oplus T_x X \oplus N_x^2 X \oplus \cdots \oplus N_x^f X$. See [24, sect. 2.2] or [9] for a quick introduction. Adopt the notation $\mathbb{FF}_1 : T_x X \to T_x X$ is the identity map.

Let $X = G/P \subset \mathbb{P}\mathbf{V}$ be generalized cominuscule. Then the only projective differential invariants of X at a point are the fundamental forms, and these are easily (in fact, pictorially) determined [24].

Let X be generalized cominuscule, let $p = [v]$, and let $v_1, \dots, v_{r-1} \in \hat{T}_p X$ be general tangent vectors. Then by the interpretation of fundamental forms in terms of osculating spaces to curves in X [13, sect. 1] a general point of the bud $\mathfrak{b}_r(X, p)$ is

$$\left[v + \sum_{k=1}^{r-1} \sum_{j_1+\cdots+j_k=r-1} \mathbb{FF}_k(v_{j_1}, \dots, v_{j_k}) \right].$$

More information on how to apply fundamental forms to parameterize linear spans of aligned schemes, i.e., spans of higher order derivatives of curves, is given in [18, sect. 3.6] and [24].

Example 4.17. When $X = v_d(\mathbb{P}W)$, and $p = [w^d]$, then for $k \le d$, the image of the transpose of the fundamental form is the full space of forms of degree k, $\mathbb{FF}_k^t(N_x^{*k}) = S^k T_x^* v_d(\mathbb{P}W)$, and is zero for $k > d$, elements of $\hat{T}_p X$ are of the form $w^{d-1}u$, and

$$\mathbb{FF}_k(w^{d-1}u_1, \dots, w^{d-1}u_k) = w^{d-k}u_1 \cdots u_k.$$

Thus a general point of the bud is of the form

$$\left[\sum_{k=0}^{r-1} \sum_{0 \le \ i_1+\cdots+i_k=r-1} w^{d-k} u_{i_1} \cdots u_{i_k} \right].$$

Example 4.18. The fundamental forms of Segre varieties are as follows: Write $A_j' = A_j/\langle a_j \rangle$. Then $T_{[a_1 \otimes \cdots \otimes \, a_k]} = A_1' \oplus \cdots \oplus A_k'$ and

$$\mathbb{FF}_l^t(N^{*l}_{[a_1 \otimes \cdots \otimes \, a_k]}) = \bigoplus_{\{i_1,\dots,i_l\} \subset [k]} A_{i_1}'^{*} \otimes \cdots \otimes A_{i_l}'^{*}.$$

In particular, for a k-factor Segre, the last nonzero fundamental form is $\mathbb{FF}_k$. The second fundamental form at $[a_1 \otimes \cdots \otimes a_k]$ is spanned by the quadrics generating the ideal of $\mathbb{P}(A_1/a_1) \sqcup \cdots \sqcup \mathbb{P}(A_k/a_k) \subset \mathbb{P}(A_1/a_1 \oplus \cdots \oplus A_k/a_k) \simeq \mathbb{P}T_{[a_1 \otimes \cdots \otimes \, a_k]} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k)$. A general point of $\mathfrak{b}_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k), [a_1 \otimes \cdots \otimes a_k])$ is of the form

$$\left[\sum_{i_1+\cdots+i_k \le r-1} a_{1,i_1} \otimes \cdots \otimes a_{k,i_k} \right],$$

where $a_{j,i_j} \in A_j$ are arbitrary elements with $a_{j,0} = a_j$.

4.3. Examples of points not in the open r-bud of generalized cominuscule varieties.

Let $X \subset \mathbb{P}\mathbf{V}$ be a generalized cominuscule variety.

Our examples are constructed from parameterized curves $x_j(t)$ in X. The general proce-

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dure to obtain the scheme to which the points degenerate as $t \to 0$ is as follows:

(1) A Zariski open subset of X has a rational parametrization. A priori r curves require $r \dim X$ different $\mathbb{C}[t]$ coefficients. However, as any scheme of length r can be embedded into a space of dimension $r-1$, we reduce to a space of dimension $r-1$.
(2) Find the ideal I of polynomial equations that defines the curves as a parametric family over $\mathbb{C} \times \mathbb{C}^{r-1}$, where the first component corresponds to the variable t.
(3) The desired scheme is given by the ideal (I, t), which may be considered as a subscheme of $\mathbb{C}^u$ for some $u \leq r-1$.

As our curves are given parametrically, the first two steps are instances of the implicitization problem. For the third, one substitutes $t = 0$ into a set of generators.

We already saw that the first possible example of a point not in the open bud is when $r = 4$. Let $r = 4$, and consider $p = \mathbb{FF}_2(v_1, v_2) + v_3$, where $v_j \in T_x X$. When $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, write $x = [a_1 \otimes b_1 \otimes c_1]$ and $v_j = a_1 \otimes b_1 \otimes c_{j+1} + a_1 \otimes b_{j+1} \otimes c_1 + a_{j+1} \otimes b_1 \otimes c_1$, so

$$p = \left[a_1 \otimes b_1 \otimes c_4 + a_1 \otimes b_4 \otimes c_1 + a_4 \otimes b_1 \otimes c_1 + \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)} \right].$$

When $X = v_3(\mathbb{P}W)$, take $x = [u^3]$, $v_1 = u^2 y$, $v_2 = u^2 z$, $v_3 = u^2 w$, and then $p = [uyz + u^2 w]$.

In the Segre case, p is in the limit 4-plane of the curve of 4-planes spanned by

$$\begin{aligned} x_0(t) &= a_1 \otimes b_1 \otimes c_1, \\ x_1(t) &= (a_1 + t a_2 + t^2 a_4) \otimes (b_1 + t b_2 + t^2 b_4) \otimes (c_1 + t c_2 + t^2 c_4), \\ x_2(t) &= (a_1 + t a_3) \otimes (b_1 + t b_3) \otimes (c_1 + t c_3), \\ x_3(t) &= (a_1 - t(a_2 + a_3)) \otimes (b_1 - t(b_2 + b_3)) \otimes (c_1 - t(c_2 + c_3)). \end{aligned}$$

If we set $b_j = c_j = a_j$, we obtain the corresponding curves in the Veronese $v_3(\mathbb{P}A)$.

Consider the affine open subset $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$ where the coordinate $a_1 \otimes b_1 \otimes c_1$ is nonzero. All of the curves belong to it. Since the coefficients in $\mathbb{C}[t]$ appearing for each j are the same for a_j, b_j, c_j, we may reduce to $\mathbb{C}^3$.

We have reduced to four curves of the form $(y_1(t), y_2(t), y_3(t))$: $(0,0,0)$, $(t, 0, t^2)$, $(0, t, 0)$, $(-t, -t, 0)$. They satisfy the equation $y_3 = y_1(y_2 + t)$, so we may focus on the first two coordinates. Hence, we have four points in the projective plane—a complete intersection of two quadrics. The generators of I are

$$y_1(y_1 - t - 2y_2), y_2(2y_1 + t - y_2).$$

Substituting $t = 0$, we obtain the annihilators of the nondegenerate quadratic form $Y_1^2 + Y_1 Y_2 + Y_2^2$ (where Y_j is dual to y_j); i.e., the limiting scheme is isomorphic to $\operatorname{Spec} \mathbb{C}[y_1, y_2]/(y_1 y_2, y_1^2 - y_2^2)$, that is, Gorenstein alignable, but not aligned. In particular, it is in the bud, but not the open bud. The Hilbert function equals $(1, 2, 1)$.

Example 4.19 (the Coppersmith–Winograd tensor). The "easy" Coppersmith–Winograd tensor is

$$T_{q,cw} := \sum_{j=1}^{q} (a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0) \in \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1}.$$

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The (second) Coppersmith–Winograd tensor generalizes the example above. It is

$$T_{q,CW} := \sum_{j=1}^{q}(a_0\otimes b_j\otimes c_j + a_j\otimes b_0\otimes c_j + a_j\otimes b_j\otimes c_0) + a_0\otimes b_0\otimes c_{q+1} + a_0\otimes b_{q+1}\otimes c_0 + a_{q+1}\otimes b_0\otimes c_0 \in \mathbb{C}^{q+2}\otimes\mathbb{C}^{q+2}\otimes\mathbb{C}^{q+2}. \tag{1}$$

It equals

$$\lim_{t\to 0}\left[\sum_{i=1}^{q}\frac{1}{t^2}(a_0+ta_i)\otimes(b_0+tb_i)\otimes(c_0+tc_i) - \frac{1}{t^3}\left(a_0+t^2\left(\sum_{j=1}^{q}a_j\right)\right)\otimes\left(b_0+t^2\left(\sum_{j=1}^{q}b_j\right)\right)\otimes\left(c_0+t^2\left(\sum_{j=1}^{q}c_j\right)\right) + \left[\frac{1}{t^3}-\frac{q}{t^2}\right](a_0+t^3a_{q+1})\otimes(b_0+t^3b_{q+1})\otimes(c_0+t^3c_{q+1})\right].$$

The Coppersmith–Winograd tensors are symmetric, $T_{q,cw}$ corresponds to the polynomial $x(y_1^2+\cdots+y_q^2)$, and $T_{q,CW}$ to the polynomial $x(xz+y_1^2+\cdots+y_q^2)$. These polynomials have symmetric ranks, respectively $2q+1$ and $2q+3$ (resp., shown in [29, 10]). In [25] we showed these agree with their tensor ranks—thus the Comon conjecture [11], that the rank and symmetric rank of a symmetric tensor agree, holds for these tensors. Moreover, since our border rank decomposition is symmetric and matches the lower bound, the border rank version of the Comon conjecture [7] holds for these tensors as well.

Since the tensor is symmetric, we may immediately reduce to $\mathbb{C}^{q+2}$ and work in the open set where $a_0 = 1$. Then in the resulting $\mathbb{C}^{q+1}$ the curves are

$$(t,0,\dots,0),\ (0,t,0,\dots,0),\dots,(0,\dots,0,t,0),(t^2,\dots,t^2,0),(0,\dots,0,t^3).$$

We notice that for all curves the following equality holds: $y_{q+1} = t^3 - (\sum_{i=1}^q y_i)t^2 + y_1y_2q - (\sum_{i=1}^q y_i)t/q + (\sum_{i=1}^q y_i^2)/q - y_1y_2$. Hence, we are reduced to the curves

$$(0,\dots,0),\quad (t,0,\dots,0),\quad (0,t,0,\dots,0),\dots,(0,\dots,0,t),\quad (t^2,\dots,t^2),$$

which satisfy the equations $y_i(y_i-t)-y_j(y_j-t)$ and $y_i(ty_i-t^2-(t-1)y_j)$ for all $i\neq j$. Hence, in the limit we obtain the Gorenstein scheme given by the annihilators of the nondegenerate quadric $Y_1^2+\cdots Y_{q-1}^2$, namely $\operatorname{Spec}\mathbb{C}[y_1,\dots,y_{q-1}]/(y_iy_j, y_i^2-y_j^2)_{1\le i<j\le q-1}$.

Remark 4.20. The schemes that we obtain are artifacts of the border rank decompositions we choose and are not intrinsic to the tensors. Even if we restrict ourselves to schemes/decompositions of minimal degree, their uniqueness is related to generalized identifiability questions.

4.4. The bud and matrix multiplication. We ask if the following strengthening of Corollary 4.4 holds.

Question 4.21. Is it true that if $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) = r$, then $M_{\langle \mathbf{n}\rangle} \in \mathfrak{b}_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C),k)$ for some $k\in\mathcal{K}$?

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We expect that a positive answer would allow one to prove $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n}^2-1$. We calculated this for $\mathbf{n}=3$. The full proof involves many cases. Here we present the two extreme cases, for arbitrary $\mathbf{n}$, to illustrate the idea.

Recall that a point of the open bud is of the form

$$\sum_{3\leq i+j+k\leq r+2} a_i\otimes b_j\otimes c_k$$

for some $a_i\in A$, $b_j\in B$, and $c_k\in C$.

Proposition 4.22. *Say $M_{\langle \mathbf{n}\rangle}\in \mathfrak{b}^0_{2\mathbf{n}^2-2}(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C),k)$. Then one cannot have any two of the three sets of vectors $\{a_1,\dots,a_{\mathbf{n}^2}\}$, $\{b_1,\dots,b_{\mathbf{n}^2}\}$, $\{c_1,\dots,c_{\mathbf{n}^2}\}$, linearly independent, nor can one have any of* $\dim\langle a_1,\dots,a_{2\mathbf{n}^2-\mathbf{n}-1}\rangle$, $\dim\langle b_1,\dots,b_{2\mathbf{n}^2-\mathbf{n}-1}\rangle$, *or* $\dim\langle c_1,\dots,c_{2\mathbf{n}^2-\mathbf{n}-1}\rangle$ *less than* $\mathbf{n}^2$.

Proof. For the first case, assume without loss of generality (by cyclic symmetry) that $a_1,\dots,a_{\mathbf{n}^2}$ and $c_1,\dots,c_{\mathbf{n}^2}$ are linearly independent. Let $\alpha\in\langle a_1,\dots,a_{\mathbf{n}^2-1}\rangle^{\perp}$ be nonzero so $\alpha(a_{\mathbf{n}^2})\neq 0$ and similarly $\gamma\in\langle c_1,\dots,c_{\mathbf{n}^2-1}\rangle^{\perp}$, $\gamma(c_{\mathbf{n}^2})\neq 0$. Consider the proper left ideal $M_{\langle \mathbf{n}\rangle}\alpha\subset B$. (Here A,B,C are all the same algebra of $\mathbf{n}\times\mathbf{n}$ matrices, but we distinguish them for clarity.) Then, viewing matrix multiplication as $C\times A\to B$, $M_{\langle \mathbf{n}\rangle}\alpha\subset B$ is the span of $b_1,\dots,b_{\mathbf{n}^2-1}$ because for each choice of $\tilde\gamma\in C$, the vector

$$(\tilde\gamma(c_{\mathbf{n}^2-1}),\dots,\tilde\gamma(c_1))\begin{pmatrix}\alpha(a_{\mathbf{n}^2}) & 0 & \cdots & 0\\ \alpha(a_{\mathbf{n}^2+1}) & \alpha(a_{\mathbf{n}^2}) & 0 & \cdots\\ \vdots & & \ddots & \\ \alpha(a_{2\mathbf{n}^2-2}) & \cdots & & \alpha(a_{\mathbf{n}^2})\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_{\mathbf{n}^2-1}\end{pmatrix}\in B$$

appears in $M_{\langle \mathbf{n}\rangle}\alpha$. On the other hand, the right ideal $\gamma M_{\langle \mathbf{n}\rangle}\subset B$ is the span of the vectors for each choice of $\tilde\alpha\in A^*$,

$$(b_{\mathbf{n}^2-1},\dots,b_1)\begin{pmatrix}\gamma(c_{\mathbf{n}^2}) & 0 & \cdots & 0\\ \gamma(c_{\mathbf{n}^2+1}) & \gamma(c_{\mathbf{n}^2}) & 0 & \cdots\\ \vdots & & \ddots & \\ \gamma(c_{2\mathbf{n}^2-2}) & \cdots & & \gamma(c_{\mathbf{n}^2})\end{pmatrix}\begin{pmatrix}\tilde\alpha(a_1)\\ \tilde\alpha(a_2)\\ \vdots\\ \tilde\alpha(a_{\mathbf{n}^2-1})\end{pmatrix},$$

which is also the span of $b_1,\dots,b_{\mathbf{n}^2-1}$, a contradiction.

For the second case, assume $\dim\langle a_1,\dots,a_{2\mathbf{n}^2-\mathbf{n}-1}\rangle<\mathbf{n}^2$. There exists a nonzero $\alpha\in\langle a_1,\dots,a_{2\mathbf{n}^2-\mathbf{n}-1}\rangle^{\perp}$. Then the nonzero left ideal $\alpha M_{\langle \mathbf{n}\rangle}$ is a subspace of $\langle c_1,\dots,c_{\mathbf{n}-1}\rangle$, but the smallest dimension of a left ideal is $\mathbf{n}$, a contradiction. ■

5. A border rank analogue of the substitution method.

We recall the Alexeev–Forbes–Tsimerman variant of the substitution method, as rephrased in [25].

Proposition 5.1 (see [1, Appendix B], [5, Chapter 6]). *Fix a basis $a_1,\dots,a_{\mathbf{a}}$ of A. Write $T=\sum_{i=1}^{\mathbf{a}} a_i\otimes M_i$, where $M_i\in B\otimes C$. Let $\mathbf{R}(T)=r$ and $M_1\neq 0$. Then there exist constants $\lambda_2,\dots,\lambda_m$, such that the tensor*

$$\tilde T := \sum_{j=2}^{m} a_j\otimes(M_j-\lambda_j M_1)\in\langle a_2,\dots,a_{\mathbf{a}}\rangle\otimes B\otimes C$$

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has rank at most $r-1$*. Moreover, if* $\operatorname{rank}(M_1) = 1$*, then for any choices of* λ_j *we have* $\mathbf{R}(\tilde{T}) \geq r-1$.

Note that this is true for any choice of basis. Here is a border rank version, which is weaker because one does not get to choose the hyperplane to project to.

Lemma 5.2. *Let* $T \in A \otimes B \otimes C$*. Then there exists a hyperplane* $H_A \subset A^*$ *such that* $\underline{\mathbf{R}}(T|_{H_A \times B^* \times C^*}) \leq \underline{\mathbf{R}}(T) - 1$.

In other words, there exists $a \in A$ *such that image* T' *of* T *under the projection* $A \otimes B \otimes C \to (A/a) \otimes B \otimes C$ *has* $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T) - 1$.

Proof. First note the statement is true for rank (AFT). Let T_t be a curve with $\lim_{t \to 0} T_t = T$. For each t the statement is true for some H_t, and by the proof in [25], one sees that the H_t will vary smoothly with t. Since projective space is compact, there is a limiting H_0 and the statement holds with $H_A = H_0$. ■

One could, in principle, apply this, alternating the roles of A, B, C, to potentially obtain border rank bounds up to $\dim A + \dim B + \dim C - 3$. It is more difficult to apply Lemma 5.2 than the usual substitution method, because one needs to check all possible hyperplanes. However, as we show, it can be applied to tensors with large symmetry groups like $M_{\langle \mathbf{n} \rangle}$.

Corollary 5.3. *Let* $M^{red}_{\langle \mathbf{n} \rangle} = M_{\langle \mathbf{n} \rangle} - \sum_j x^1_{\mathbf{n}} \otimes y^{\mathbf{n}}_j \otimes z^j_1$ *be a reduced matrix multiplication operator. Then* $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq \underline{\mathbf{R}}(M^{red}_{\langle \mathbf{n} \rangle}) + 1$.

Proof. There are $\mathbf{n}$ nonzero orbits in A under $G_{M_{\langle \mathbf{n} \rangle}}$, namely the matrices of rank r, $1 \leq r \leq \mathbf{n}$. Say the element x of A that works in Lemma 5.2 has rank r. If we act on it by an element of $G_{M_{\langle \mathbf{n} \rangle}}$, the new element still works, so it will work for points in the orbit closure $\overline{G_{M_{\langle \mathbf{n} \rangle}} \cdot x}$, which contain the rank one elements. Finally, all rank one elements are equivalent to $x^1_{\mathbf{n}}$. ■

6. A new lower bound for the border rank of matrix multiplication.

Theorem 6.1. *When* $\mathbf{n} \geq 3$, $\underline{\mathbf{R}}(M^{red}_{\langle \mathbf{n} \rangle}) \geq 2\mathbf{n}^2 - \mathbf{n}$.

When $\mathbf{n} = 2$, it was shown in [28] that $\underline{\mathbf{R}}(M^{red}_{\langle 2 \rangle}) = 5$.

Theorem 6.1 combined with Corollary 5.3 implies the next theorem.

Theorem 6.2. *Let* $\mathbf{n} \geq 3$*; then* $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq 2\mathbf{n}^2 - \mathbf{n} + 1$.

Remark 6.3. The result is also true for $\mathbf{n} = 2$, where it is optimal, as was shown in [21, 16].

Remark 6.4. The state of the art in other small cases is $16 \leq \underline{\mathbf{R}}(M_{\langle 3 \rangle}) \leq 20$ (the upper bound appears in [33]), and $29 \leq \underline{\mathbf{R}}(M_{\langle 4 \rangle}) \leq 46$ (with the upper bound in the unpublished [34]).

Proof of Theorem 6.1. We use the Koszul flattenings defined in [27] which were used to prove $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq 2\mathbf{n}^2 - \mathbf{n}$. Write $M^{red}_{\langle \mathbf{n} \rangle} \in \mathbb{C}^{\mathbf{n}^2 - 1} \otimes B \otimes C = \tilde{A} \otimes B \otimes C$. Consider for any $T = \sum t^{ijk} a_i \otimes b_j \otimes c_k \in \tilde{A} \otimes B \otimes C$, the Koszul flattening $T^{\wedge \mathbf{n}-1}_{\tilde{A}} : \Lambda^{\mathbf{n}-1} \tilde{A} \otimes B^* \to \Lambda^{\mathbf{n}} \tilde{A} \otimes C$ given by

$$f_1 \wedge \cdots \wedge f_{n-1} \otimes \beta \mapsto \sum \beta(b_j) t^{ijk} a_i \wedge f_1 \wedge \cdots \wedge f_{n-1} \otimes c_k.$$

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For matrix multiplication and for the reduced matrix multiplication, this map factors $B = V^*{\otimes}W$ and $C = W^*{\otimes}U$, where $U, V, W = \mathbb{C}^{\mathbf{n}}$ to $(M_{\langle 1,\mathbf{n},\mathbf{n}\rangle})_A^{\wedge \mathbf{n}-1}{\otimes}\mathrm{Id}_{W^*}$ and $(M^{red}_{\langle 1,\mathbf{n},\mathbf{n}\rangle})_{\tilde{A}}^{\wedge \mathbf{n}-1}{\otimes}\mathrm{Id}_{W^*}$, respectively, where $(M^{red}_{\langle 1,\mathbf{n},\mathbf{n}\rangle})_{\tilde{A}}^{\wedge \mathbf{n}-1} : \Lambda^{\mathbf{n}-1}\tilde{A}{\otimes}V \to \Lambda^{\mathbf{n}}\tilde{A}{\otimes}U$. As in [27], we obtain the best result by restricting to an $A' := \mathbb{C}^{2n-1} \subset \tilde{A}$. Define the map

$$\begin{aligned} \phi : \tilde{A} &\to A', \\ x^i_j &\mapsto e_{i+j-1}. \end{aligned}$$

Since $(a{\otimes}b{\otimes}c)^{\wedge(\mathbf{n}-1)}_{\mathbb{C}^{2n-1}}$ has rank $\binom{2\mathbf{n}-2}{\mathbf{n}-1}$ (its image is $a \wedge \Lambda^{\mathbf{n}-1}(\mathbb{C}^{2n-1}/a){\otimes}c$), it will suffice to prove, for $\mathbf{n} \geq 3$, that

$$\frac{\mathrm{rank}(M^{red}_{\langle 1,\mathbf{n},\mathbf{n}\rangle})^{\wedge \mathbf{n}-1}_{A'}}{\binom{2\mathbf{n}-2}{\mathbf{n}-1}} \geq 2\mathbf{n} - 1.$$

We claim that $\mathrm{rank}(M^{red}_{\langle 1,\mathbf{n},\mathbf{n}\rangle})^{\wedge \mathbf{n}-1}_{A'} = \mathbf{n}\binom{2\mathbf{n}-1}{\mathbf{n}} - 1$, which will prove the result when $\mathbf{n} \geq 3$.

Write $e_S = e_{s_1} \wedge \cdots \wedge e_{s_{\mathbf{n}-1}}$, where $S \subset [2\mathbf{n}-1]$ has cardinality $\mathbf{n}-1$. Our map is

$$e_S{\otimes}v_k \mapsto \sum_{\{m|(m,k)\neq(\mathbf{n},1)\}} \phi(u^m{\otimes}v_k) \wedge e_S{\otimes}u_m = \sum_{\{m|(m,k)\neq(\mathbf{n},1)\}} e_{m+k-1} \wedge e_S{\otimes}u_m.$$

Index a basis of the source by pairs (S, k), with $k \in [\mathbf{n}]$, and the target by (P, l) where $P \subset [2\mathbf{n}-1]$ has cardinality $\mathbf{n}$ and $l \in [\mathbf{n}]$. We define an order relation on the target basis vectors in the following way. For (P_1, l_1) and (P_2, l_2), set $l = \min\{l_1, l_2\}$, and declare $(P_1, l_1) < (P_2, l_2)$ if and only if

(1) in lexicographic order, the set of l minimal elements of P_1 is strictly after the set of l minimal elements of P_2 (i.e., the smallest element of P_2 is smaller than the smallest of P_1 or they are equal and the second smallest of P_2 is smaller than or equal, etc., up to lth), or
(2) the l minimal elements in P_1 and P_2 are the same, and $l_1 < l_2$,
(3) the l minimal elements in P_1 and P_2 are the same, and $l_1 = l_2$, and the set of $\mathbf{n} - l$ tail elements of P_1 are after the set of $\mathbf{n} - l$ tail elements of P_2.

The third ordering is actually irrelevant—any breaking of a tie for the first two will lead to an upper-triangular matrix. Note that $(\{\mathbf{n}, \dots, 2\mathbf{n}-1\}, 1)$ is the unique minimal element for this relation and $([\mathbf{n}], \mathbf{n})$ is the unique maximal element. Note further that

$$e_{n+1} \wedge \cdots \wedge e_{2\mathbf{n}-1}{\otimes}u_n \mapsto e_n \wedge \cdots \wedge e_{2\mathbf{n}-1}{\otimes}v_1,$$

i.e., that

$$(\{\mathbf{n}+1, \dots, 2\mathbf{n}-1\}, \mathbf{n}) \mapsto (\{\mathbf{n}, \dots, 2\mathbf{n}-1\}, 1).$$

We will prove the claim by showing that the image is the span of all basis elements (P, l) except the maximal element $([\mathbf{n}], \mathbf{n})$. We work by induction using the relation; the base case that $(\{\mathbf{n}, \dots, 2\mathbf{n}-1\}, 1)$ is in the image has been established. Let (P, l) be any basis element other than the maximal, and assume all (P', l') with $(P', l') < (P, l)$ have been shown to be in

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the image. Write $P = (p_1, \dots, p_{\mathbf{n}})$ with $p_i < p_{i+1}$. Consider the image of $(P\backslash\{p_l\}, 1 + p_l - l)$, which is

$$\sum_{\{m|(m,1+p_l-l)\neq(\mathbf{n},1)\}} \phi(u^m \otimes v_{1+p_l-l}) \wedge e_{P\backslash\{p_l\}} \otimes u_m = \sum_{\{m|(m,1+p_l-l)\neq(\mathbf{n},1)\}} e_{p_l-l+m} \wedge e_{P\backslash\{p_l\}} \otimes u_m.$$

In particular, taking $m = l$ we see (P, l) is among the summands, as long as (P, l) is not the maximal element. If $m < l$, the contribution to the summand is a (P', m) where the first m terms of P' equal the first of P, so by condition (2), $(P', m) < (P, l)$. If $m > l$, the summand is a (P'', m) where the first $l - 1$ terms of P and P'' agree, and the lth terms are, respectively, p_l and $p_l - l + m$, so by condition (1) $(P'', m) < (P, l)$. ■

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