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A universal dimension formula for complex simple Lie algebras

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Abstract

We present a universal formula for the dimension of the Cartan powers of the adjoint representation of a complex simple Lie algebra (i.e., a universal formula for the Hilbert functions of homogeneous complex contact manifolds), as well as several other universal formulas. These formulas generalize formulas of Vogel and Deligne and are given in terms of rational functions where both the numerator and denominator decompose into products of linear factors with integer coefficients. We discuss consequences of the formulas including a relation with Scorza varieties.

Keywords: Universal Lie algebra; Scorza variety; Homogeneous complex contact manifold

1. Statement of the main result

Vogel [17] defined a tensor category $\mathcal{D}'$ intended to be a model for a universal simple Lie algebra. His motivation came from knot theory, as $\mathcal{D}'$ was designed to surject

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onto the category of Vassiliev invariants. While Vogel's work remains unfinished (and unpublished), it already has consequences for representation theory.

Let $\mathfrak{g}$ be a complex simple Lie algebra. Vogel derived a universal decomposition of $S^2\mathfrak{g}$ into (possibly virtual) Casimir eigenspaces, $S^2\mathfrak{g} = \mathbb{C} \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma)$ which turns out to be a decomposition into irreducible modules. If we let $2t$ denote the Casimir eigenvalue of the adjoint representation (with respect to some invariant quadratic form), these modules, respectively, have Casimir eigenvalues $4t - 2\alpha, 4t - 2\beta, 4t - 2\gamma$, which we may take as the definitions of α, β, γ. Vogel showed that $t = \alpha + \beta + \gamma$. He then went on to find Casimir eigenspaces $Y_3(\alpha), Y_3(\beta), Y_3(\gamma) \subset S^3\mathfrak{g}$ with eigenvalues $6t - 6\alpha, 6t - 6\beta, 6t - 6\gamma$ (which again turn out to be irreducible), and computed their dimensions through difficult diagrammatic computations and the help of Maple [17]:

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma},$$

$$\dim Y_2(\alpha) = -\frac{t(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)(3\alpha - 2t)}{\alpha^2\beta\gamma(\alpha - \beta)(\alpha - \gamma)}.$$

$$\dim Y_3(\alpha) = -\frac{t(\alpha - 2t)(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)(t + \beta - \alpha)(t + \gamma - \alpha)(5\alpha - 2t)}{\alpha^3\beta\gamma(\alpha - \beta)(\alpha - \gamma)(2\alpha - \beta)(2\alpha - \gamma)}$$

and the formulas for $Y_2(\beta), Y_2(\gamma)$ and $Y_3(\beta), Y_3(\gamma)$ are obtained by permuting α, β, γ. These formulas suggest a completely different perspective from the usual description of the simple Lie algebras by their root systems and the Weyl dimension formula that can be deduced for each particular simple Lie algebra. The work of Vogel raises many questions. In particular, what remains of these formulas when we go to higher symmetric powers? If such formulas do exist in general, do we need to go to higher and higher algebraic extensions to state them, as Vogel suggests? Vogel describes modules in the third tensor power of the adjoint representation that require an algebraic extension for their dimension formulas.

For the exceptional series of simple Lie algebras, explicit computations of Deligne, Cohen and de Man showed that the decompositions of the tensor powers are well-behaved up to degree 4, after which modules appear whose dimensions are *not* given by rational functions whose numerator and denominator are products of linear factors with integer coefficients (see [7,13] for proofs of such types of formulas). In both the works of Vogel and Deligne et al., problems arise when there are different irreducible modules appearing in a Schur component with the same Casimir eigenvalue.

In this paper, we show that some of the phenomena observed by Vogel and Deligne do persist in all degrees. Let α_0 denote the highest root of $\mathfrak{g}$, once we have fixed a Cartan subalgebra and a set of positive roots.

Theorem 1.1. *Use Vogel's parameters α, β, γ as above. The kth symmetric power of $\mathfrak{g}$ contains three (virtual) modules $Y_k(\alpha), Y_k(\beta), Y_k(\gamma)$ with Casimir eigenvalues $2kt -$*

$(k^2-k)\alpha, 2kt-(k^2-k)\beta, 2kt-(k^2-k)\gamma$. *Using binomial coefficients defined by* $\binom{y+x}{y}=(1+x)\cdots(y+x)/y!$, *we have*

$$\dim Y_k(\alpha)=\frac{t-\left(k-\frac{1}{2}\right)\alpha}{t+\frac{\alpha}{2}}\frac{\binom{-\frac{2t}{\alpha}-2+k}{k}\binom{\frac{\beta-2t}{\alpha}-1+k}{k}\binom{\frac{\gamma-2t}{\alpha}-1+k}{k}}{\binom{-\frac{\beta}{\alpha}-1+k}{k}\binom{-\frac{\gamma}{\alpha}-1+k}{k}},$$

and $\dim Y_k(\beta)$ *and* $\dim Y_k(\gamma)$ *are obtained by exchanging the role of* α *with* β *and* γ, *respectively*.

The modules $Y_k(\beta), Y_k(\gamma)$ are described in Section 6. For $Y_k(\alpha)$, we have the following refinement:

Theorem 1.2. *Parametrize the complex simple Lie algebras as follows*:

Series	*Lie algebra*	α	β	γ
SP	$\mathfrak{sp}_{2n}$	-2	1	$n+2$
SL	$\mathfrak{sl}_n$	-2	2	n
SO	$\mathfrak{so}_n$	-2	4	$n-4$
EXC		-2	$a+4$	$2a+4$
	$\mathfrak{sl}_3$	-2	3	2
	$\mathfrak{g}_2$	-2	$10/3$	$8/3$
	$\mathfrak{so}_8$	-2	4	4
	$\mathfrak{f}_4$	-2	5	6
	$\mathfrak{e}_6$	-2	6	8
	$\mathfrak{e}_7$	-2	8	12
	$\mathfrak{e}_8$	-2	12	20
$F3_r$		-2	a	$a(r-2)+4$

Then $Y_k(\alpha)$ *is the kth Cartan power* $\mathfrak{g}^{(k)}$ *of* $\mathfrak{g}$ (*the module with highest weight* $k\alpha_0$) *and*

$$\dim\mathfrak{g}^{(k)}=\frac{\beta+\gamma-3+2k}{\beta+\gamma-3}\frac{\binom{\beta+\gamma/2-3+k}{k}\binom{\gamma+\beta/2-3+k}{k}\binom{\beta+\gamma-4+k}{k}}{\binom{-1+\beta/2+k}{k}\binom{-1+\gamma/2+k}{k}}.$$

In the exceptional series *EXC*, we have $a=-1,-2/3,0,1,2,4,8$. Here $F3_r$ denotes the two-parameter series of Lie algebras in the generalized third row of Freudenthal's magic chart, $\mathfrak{g}_r(\mathbb{H},\mathbb{A})$ with $a=1,2,4$ and $r\geqslant 3$, which contains $\mathfrak{sp}_{2r}, \mathfrak{sl}_{2r}, \mathfrak{so}_{4r}$, and $\mathfrak{e}_7$ when $r=3$ [13]. We call $F3_3$ the *subexceptional series*.

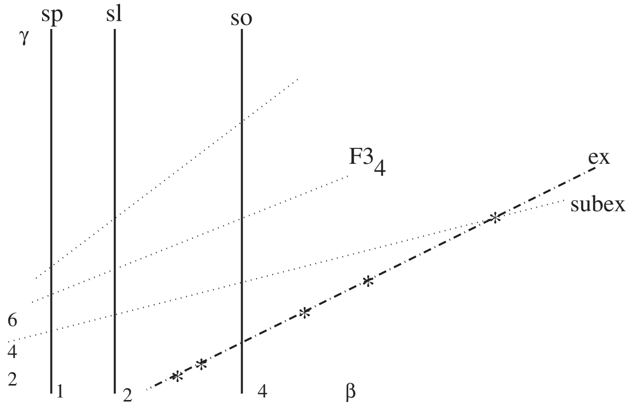


Fig. 1. Vogel's plane.

The parameters (α, β, γ) may be thought of as defining a point in $\mathbb{P}^2/\mathfrak{S}_3$ which we refer to as *Vogel's plane* (Fig. 1). We say a collection of points lie on a *line* in Vogel's plane if some lift of them to $\mathbb{P}^2$ is a colinear set of points. The classical series $\mathfrak{sl}$, $\mathfrak{so}$, $\mathfrak{sp}$ all lie on lines by the description above and one can even make the points of $\mathfrak{so}$ and $\mathfrak{sp}$ lie on the same line. The algebras in the exceptional series all lie on a line, as do the algebras in each of the generalized third rows of Freudenthal's magic chart. Through each classical simple Lie algebra there are an infinite number of lines with at least three points. Distinguished among these are the above-mentioned lines. For each of these, there are natural inclusions of the Lie algebras as one travels north-east along the line.

Dotted diagonal lines correspond to $F3_r$.

Remark 1.3. The reason $\mathfrak{so}$ and $\mathfrak{sp}$ are split into two different lines is that we require $Y_k(\alpha)$ to be the Cartan powers of the adjoint representation. The formula for $\dim \mathfrak{g}^{(k)}$ applied to the $\mathfrak{sp}$ series situated as $(-2, 4, -2n)$ yields the dimensions of the modules $Y_k(\gamma)$.

1.1. Overview

In §2–4 we prove the main result, which is based on a careful analysis of the five step grading of a simple Lie algebra defined by the highest root. In §5 we show how this relates to other $\mathbb{Z}$ and $\mathbb{Z}_2$-gradings and give a dimension formula for $\dim \mathfrak{g}^{(k)} Y_2(\beta)^{(l)}$. In §6 we describe the modules $Y_k(\beta)$, $Y_k(\gamma)$ explicitly. We show that the highest weight of $Y_k(\beta)$ is the sum of k orthogonal long roots, and give geometric interpretations of them related to Scorza varieties. We conclude with an infinite series of dimension formulas for the Cartan powers of the $Y_k(\beta)$. These formulas show that the modules $Y_k(\alpha)$ and $Y_k(\beta)$ should be considered as universal in a very strong sense. Giving a precise meaning to that last sentence is an interesting open problem.

1.2. Further questions and comments

Remarkably, the numbers β and γ also appear in [12] in connection with the McKay correspondence. The numbers h, h' are exponents of $\mathfrak{g}$. For $\mathfrak{g}$ simply laced, they coincide with the intermediate exponents of the functions $z(t)$ in [12] having a linear factor. Why?

The formulas above, in addition to having zeros and poles, have indeterminacy loci. For example, the point corresponding to $\mathfrak{so}_8$ is in the indeterminacy locus of $\dim Y_2(\beta)$. For $\mathfrak{so}_8$, $Y_2(\beta) \oplus Y_2(\gamma)$ is the sum of the three isomorphic 35 dimensional representations $2\omega_1, 2\omega_3, 2\omega_4$. We obtain $\dim Y_2(\beta) = 105$ (and $\dim Y_2(\gamma) = 0$) when considering $\mathfrak{so}_8$ as a member of the exceptional series and $\dim Y_2(\beta) = 70$ (and $\dim Y_2(\gamma) = 35$) when considering it as an element of the orthogonal series. The same phenomenon occurs for $\mathfrak{sl}_2$ which is also in the indeterminacy loci of $\dim Y_2(\beta), \dim Y_2(\gamma)$. While these remarks apply already to Vogel's results (although we are unaware of them being pointed out before) with the increasing number of points in the indeterminacy loci as k becomes large, it might be interesting to address this issue in more detail.

We remark that, for k sufficiently negative, the formulas above make sense and give rise to dimensions of virtual modules. For example, in the exceptional and subexceptional series, if one sets $K = 2t/\alpha + 1 - k$ then $\dim Y_K(\alpha) = -\dim Y_k(\alpha)$ and the dimensions for k between -1 and $2t/\alpha$ are zero. Similar phenomena occur for the classical series.

Viewing the same equations with a different perspective, we mention the work of Cvitanovic [4,5], El Houari [9,10] and Angelopolous [1] which preceeded the work of Vogel and Deligne. Their works contain calculations similar to Vogel's, but with a different goal: they use the fact that dimensions of vector spaces are integers to classify complex simple Lie algebras, and to organize them into series, using Casimirs and invariants of the symmetric algebra to obtain diophantine equations.

If one restricts to the exceptional line, Cohen and deMan have observed that (just using a finite number of dimension formulae), the only value of a nontrivially yielding non-negative integers is, with our parametrization, $a = 6$. We account for this in [15] with a Lie algebra which is intermediate between $\mathfrak{e}_7$ and $\mathfrak{e}_8$. It is an exceptional analogue of the *odd symplectic Lie algebras*. In fact, the odd symplectic groups appear to satisfy the formulas above when one allows γ to be a half-integer in the symplectic line. What other parameter values yield integers in all the formulas? Do the intermediate Lie algebras considered in [15] belong in Vogel's plane?

2. The role of the principal $\mathfrak{sl}_2$

2.1. How to use the Weyl dimension formula

A vector $X_{\alpha_0} \in \mathfrak{g}_{\alpha_0}$ belongs to the minimal (non-trivial) nilpotent orbit in $\mathfrak{g}$. We can choose $X_{-\alpha_0} \subset \mathfrak{g}_{-\alpha_0}$ such that

$$(X_{\alpha_0}, X_{-\alpha_0}, H_{\alpha_0} = [X_{\alpha_0}, X_{-\alpha_0}])$$

is a $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, generating a subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{sl}_2^*$. This is the *principal* $\mathfrak{sl}_2$. The semi-simple element H_{α_0} defines a grading on $\mathfrak{g}$ according to the eigenvalues of $ad(H_{\alpha_0})$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

The line $\mathfrak{g}_2$ (resp. $\mathfrak{g}_{-2}$) is generated by X_{α_0} (resp. $X_{-\alpha_0}$). The subalgebra $\mathfrak{g}_0$ is reductive, and splits into the sum of the line generated by H_{α_0} and the centralizer $\mathfrak{h}$ of the $\mathfrak{sl}_2$-triple. The $\mathfrak{h}$-module $\mathfrak{g}_1$ is the sum of the root spaces $\mathfrak{g}_\beta$, where β belongs to the set Φ_1 of positive roots such that $\beta(H_{\alpha_0}) = 1$. Its dimension is twice the dual Coxeter number of $\mathfrak{g}$, minus four [11].

Let ρ denote the half-sum of the positive roots. By the Weyl dimension formula,

$$\dim \mathfrak{g}^{(k)} = \frac{(\rho + k\alpha_0, \alpha_0)}{(\rho, \alpha_0)} \prod_{\beta \in \Phi_1} \frac{(\rho + k\alpha_0, \beta)}{(\rho, \beta)}.$$

We thus need to analyze the distribution of the values of (ρ, β) for $\beta \in \Phi_1$.

2.2. The $\mathbb{Z}_2$-grading

To do this, we slightly modify our grading of $\mathfrak{g}$. Let $V = \mathfrak{g}_1$ be considered as an irreducible $\mathfrak{h} \times \Gamma$-module, where Γ is the automorphism group of the Dynkin diagram of $\mathfrak{g}$. As an $\mathfrak{h}$-module, V is irreducible except in the case $\mathfrak{g} = \mathfrak{sl}_n$ where $\mathfrak{h} = \mathfrak{gl}_{n-2}$ and $V = V_{\omega_1} \oplus V_{\omega_{n-3}}$ as an $\mathfrak{h}$-module.

The space V is endowed with a natural symplectic form ω defined, up to scale, by the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2$. Thus, for each root $\beta \in \Phi_1$, $\alpha_0 - \beta$ is again a root in Φ_1. Consider $U = \mathfrak{g}_{\beta-\alpha_0} \oplus \mathfrak{g}_\beta \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. U is stable under the adjoint action of $\mathfrak{sl}_2^*$, and is a copy of the natural two-dimensional $\mathfrak{sl}_2$-module. As a $\mathfrak{sl}_2^* \times \mathfrak{h}$-module, we thus get a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}} = \mathfrak{sl}_2^* \times \mathfrak{h} \oplus U \otimes V.$$

The Lie bracket defines an equivariant map

$$\begin{array}{ccccc} \wedge^2(U \otimes V) = & S^2U \otimes \wedge^2 V & \oplus & \wedge^2 U \otimes S^2 V \\ & \downarrow {\scriptstyle id \otimes \omega} & & \downarrow {\scriptstyle \theta} \\ & \mathfrak{sl}_2^* & \oplus & \mathfrak{h}. \end{array}$$

Here we use the natural identifications $\mathfrak{sl}_2^* = S^2U$, and $\wedge^2 U = \mathbb{C}$. Moreover, the fact that $\mathfrak{h}$ preserves the symplectic form ω on V implies that the image of $\mathfrak{h}$ in $End(V) \simeq V \otimes V$ must be contained in $S^2V \simeq S^2V^*$. The map θ, up to scale, is dual to that inclusion.

Series	$\mathfrak{g}$	$\mathfrak{h}$	V	v
SL	$\mathfrak{sl}_n$	$\mathfrak{gl}_{n-2}$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	$n-2$
SO	$\mathfrak{so}_n$	$\mathfrak{sl}_2 \times \mathfrak{so}_{n-4}$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-4$
SP	$\mathfrak{sp}_{2n}$	$\mathfrak{sp}_{2n-2}$	$\mathbb{C}^{2n-2}$	$n-1$
EX	$\mathfrak{g}_2$	$\mathfrak{sl}_2$	$S^3\mathbb{C}^2$	2
	$\mathfrak{so}_8$	$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	4
	$\mathfrak{f}_4$	$\mathfrak{sp}_6$	$\wedge^{\langle 3\rangle}\mathbb{C}^6$	7
	$\mathfrak{e}_6$	$\mathfrak{sl}_6$	$\wedge^3\mathbb{C}^6$	10
	$\mathfrak{e}_7$	$\mathfrak{so}_{12}$	$V_{\omega_6} = \Delta_+$	16
	$\mathfrak{e}_8$	$\mathfrak{e}_7$	$V_{\omega_7} = \mathfrak{z}_2(\mathbb{O})$	28

Note that $\mathfrak{sl}_2^* \times \mathfrak{h}$ is a reductive subalgebra of the maximal rank of $\mathfrak{g}$. We choose a Cartan subalgebra of $\mathfrak{g}$, by taking the direct sum of $\mathbb{C}H_{\alpha_0}$, and a Cartan subalgebra of $\mathfrak{h}$. The roots of $\mathfrak{g}$ will then be the root $\alpha_0 = 2\omega_0$ of $\mathfrak{sl}_2^*$, the roots of $\mathfrak{h}$ and the weights of $U \otimes V$, i.e., the sums $\pm\omega_0 + \mu$ with μ a weight of V. We can choose a set of positive roots of $\mathfrak{h}$, and if we choose the direction of α_0 to be very positive, the positive roots of $\mathfrak{g}$ will be α_0, the positive roots of $\mathfrak{h}$, and the weights $\omega_0 + \mu$ for μ any weight of V. Note that since V is symplectic, the sum of these weights must be zero. Write $2v = \dim V$, we have

$$2\rho = 2\rho_{\mathfrak{h}} + (1+v)\alpha_0$$

and

$$\Phi_1 = \{\omega_0 + \mu \mid \mu \text{ a weight of } V\}.$$

The set of simple roots of $\mathfrak{g}$ is easily described. If $\mathfrak{g}$ is not of type A, denote the highest weight of the irreducible $\mathfrak{h}$-module V by χ so that its lowest weight is $-\chi$. For type A, denote the highest weights by χ_1, χ_2, The simple roots of $\mathfrak{g}$ are the simple roots of $\mathfrak{h}$ union $\omega_0 - \chi$ ($\omega_0 - \chi_1, \omega_0 - \chi_2$ for type A). In particular the Dynkin diagram of $\mathfrak{g}$ is the diagram of $\mathfrak{h}$ with a vertex attached to the simple roots β of $\mathfrak{g}$ such that $(\chi, \beta) \neq 0$, with the obvious analogue attaching two vertices for type A.

Remark. If we had chosen the directions of $\mathfrak{h}$ to be much more positive than that of $\mathfrak{sl}_2^*$, we would have obtained a different set of positive roots and, *except for* $\mathfrak{g} = \mathfrak{g}_2$, the highest root $\tilde{\alpha}$ of $\mathfrak{h}$ would have been the highest root of $\mathfrak{g}$ (here we suppose that $\mathfrak{h}$ itself is simple; otherwise we can take the highest root of any simple factor of $\mathfrak{h}$). We suppose in the sequel that we are not in type $\mathfrak{g}_2$: then α_0 and $\tilde{\alpha}$, considered as roots of $\mathfrak{g}$, are both long.

For type C, the root $\omega_0+\chi$ is short and it is long in all other cases. This is because, except in type A which can be checked separately, $\alpha_0 = k\omega_i$ for some fundamental weight ω_i, and $\omega_0+\chi$, being the second highest root, must equal $k\omega_i - \alpha_i$. But then,

$$(k\omega_i - \alpha_i, k\omega_i - \alpha_i) = (k\omega_i, k\omega_i) + (1-k)(\alpha_i, \alpha_i).$$

We conclude that $\omega_0+\chi$ is long iff the adjoint representation is fundamental, i.e., iff we are not in type C. Then

$$(\omega_0+\chi, \omega_0+\chi) = (\alpha_0, \alpha_0) = (\tilde{\alpha}, \tilde{\alpha}) = \frac{4}{3}(\chi, \chi).$$

Another interesting relation can be deduced from the fact that for any simple root α of $\mathfrak{g}$, we have $(2\rho, \alpha) = (\alpha, \alpha)$, since ρ is the sum of the fundamental weights. Applying this to $\alpha = \omega_0 - \chi$, we get

$$(\chi + 2\rho_h, \chi) = \frac{2v+1}{4}(\alpha_0, \alpha_0).$$

Note that the scalar form here is the Killing form of $\mathfrak{g}$, more precisely the dual of its restriction to the Cartan subalgebra. Restricted to the duals of the Cartan subalgebras of $\mathfrak{sl}_2^*$ or $\mathfrak{h}$, we can compare it to their Killing forms. Suppose that $\mathfrak{h} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_m$, and $V = V_1 \otimes \cdots \otimes V_m$ for some $\mathfrak{h}_i$-modules V_i.

To simplify notation in the calculations that follow we use the normalization that the Casimir eigenvalue of every simple Lie algebra is 1, i.e., we use for invariant quadratic form the Killing form $K(X, Y) = \mathrm{trace}(ad(X) \circ ad(Y))$.

Then for $X \in \mathfrak{sl}_2^*$ and $Y \in \mathfrak{h}_i$, we have

$$\mathrm{trace}_{\mathfrak{g}} ad(X)^2 = \mathrm{trace}_{\mathfrak{sl}_2^*} ad(X)^2 + 2v\,\mathrm{trace}_U X^2 = \left(1 + \frac{v}{2}\right) \mathrm{trace}_{\mathfrak{sl}_2^*} ad(X)^2,$$

$$\mathrm{trace}_{\mathfrak{g}} ad(Y)^2 = \mathrm{trace}_{\mathfrak{h}_i} ad(Y)^2 + 2\frac{\dim V}{\dim V_i} \mathrm{trace}_{V_i} Y^2 = (1 + 4v e_{V_i}) \mathrm{trace}_{\mathfrak{h}} ad(Y)^2,$$

where e_{V_i} is related to the Casimir eigenvalue c_{V_i} of V_i by the identity $e_{V_i} = \frac{c_{V_i}}{\dim \mathfrak{h}_i}$. Taking duals, we deduce that

$$\begin{aligned}(\alpha_0, \alpha_0) &= \frac{2}{v+2}(\alpha_0, \alpha_0)_{\mathfrak{sl}_2^*} = \frac{1}{v+2},\\ (\tilde{\alpha}, \tilde{\alpha})_{\mathfrak{h}} &= \frac{4}{3}(\chi, \chi)_{\mathfrak{h}} = \frac{1 + 4v e_V}{v+2}.\end{aligned}$$

Note that the dual Coxeter number of a simple Lie algebra is the Casimir eigenvalue of the Lie algebra divided by the length of the longest root. We conclude that the dual Coxeter number $\check{h}$ of $\mathfrak{g}$ is $\check{h} = v + 2$, while the dual Coxeter number of $\mathfrak{h}$, which we denote by h, is equal to $\frac{1+4ve_V}{v+2}$. Remember that the normalization of the Killing form is such that $(\tilde{\alpha} + 2\rho_{\mathfrak{h}}, \tilde{\alpha})_{\mathfrak{h}} = 1$, so that $(2\rho_{\mathfrak{h}}, \tilde{\alpha})_{\mathfrak{h}} = (h-1)(\tilde{\alpha}, \tilde{\alpha})_{\mathfrak{h}}$, and thus $(2\rho_{\mathfrak{h}}, \tilde{\alpha}) = (h-1)(\tilde{\alpha}, \tilde{\alpha})$ as well.

3. The Casimir eigenvalues of $S^2\mathfrak{g}$

3.1. A nontrivial component in the symmetric square of $\mathfrak{g}$

Vogel proved that $S^2\mathfrak{g}$ can contain at most four Casimir eigenspaces (allowing the possibility of zero, or even virtual eigenspaces). Two irreducible components are obvious: the Cartan square, whose highest, weight is $2\alpha_0$, and the trivial line generated by the Killing form. We identify, for $\mathfrak{g}$ not of type A_1 (i.e., $\mathfrak{h} \neq 0$), another component.

Proposition 3.1. *The symmetric square $S^2\mathfrak{g}$ has a component $Y_2(\beta)$ of highest weight $\alpha_0 + \tilde{\alpha}$.*

Proof. From our $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{sl}_2 \times \mathfrak{h} \oplus U \otimes V$, we deduce that

$$\begin{aligned} S^2\mathfrak{g} = {} & S^2\mathfrak{sl}_2 \oplus S^2\mathfrak{h} \oplus (S^2U \otimes S^2V) \oplus (\wedge^2 U \otimes \wedge^2 V) \oplus (\mathfrak{sl}_2 \otimes \mathfrak{h}) \\ & \oplus (\mathfrak{sl}_2 \otimes U \otimes V) \oplus (U \otimes \mathfrak{h} \otimes V). \end{aligned}$$

All the weights here are of the form $k\omega_0 + \mu$ for μ in the weight lattice of $\mathfrak{h}$, and we will call the integer k the *level* of the weight. The maximal level is four, the unique weight of level four is $2\alpha_0$, the highest weight of $S^2\mathfrak{sl}_2$. The corresponding weight space, of dimension one, generates the Cartan square of $\mathfrak{g}$.

We will check that once we have suppressed the weights of the Cartan square with their multiplicities, the highest remaining weight is $\alpha_0 + \tilde{\alpha}$, which has level two.

At level three, we only get weights coming from $\mathfrak{sl}_2 \otimes U \otimes V$. More precisely, let e, f be a basis of U diagonalizing our Cartan subalgebra, in such a way that the semisimple element H of our $\mathfrak{sl}_2$-triple has eigenvalues 1 on e, -1 on f, while $X = f^* \otimes e$ and $Y = e^* \otimes f$. Then a weight vector of level three in $\mathfrak{sl}_2 \otimes U \otimes V$ is of the form $X \otimes e \otimes v$, for some weight vector $v \in V$, and such a weight vector is contained in $\mathfrak{g}.X^2 \subset S^2\mathfrak{g}$, and hence in the Cartan square of $\mathfrak{g}$. It is equal, up to a nonzero constant, to $(f \otimes v).X^2$. We conclude that all weight vectors of level three belong to the Cartan square of $\mathfrak{g}$.

We turn to level two. First observe that $\alpha_0 + \tilde{\alpha}$ has multiplicity two inside $S^2\mathfrak{g}$. It is the highest weight of $\mathfrak{sl}_2 \otimes \mathfrak{h}$, which appears twice in the decomposition of $S^2\mathfrak{g}$

above: once as such, and once in a slightly more hidden way, as a component of $S^2U \otimes S^2V$. Indeed, recall that S^2U and $\mathfrak{sl}_2$ are equal, and that we defined a nontrivial map $\theta : S^2V \to \mathfrak{h}$. We check that $\alpha_0 + \tilde{\alpha}$ has only multiplicity one inside the Cartan square of $\mathfrak{g}$, and our claim will follow.

Note that this Cartan square is $U(\mathfrak{n}_-)X^2$, where $\mathfrak{n}_- \subset \mathfrak{g}$ is the subalgebra generated by the negative root spaces and $U(\mathfrak{n}_-)$ its universal enveloping algebra. As a vector space, this algebra is generated by monomials on vectors of negative weight, hence of nonpositive level. How can we go from X^2, which is of level four, to some vector of level two? We have to apply a vector of level -2, or twice a vector of level -1. For the first case, the only possible vector is Y, which maps X^2 to $XH \in S^2\mathfrak{sl}_2$. For the second case, we first apply some vector $f \otimes v$, with $v \in V$: this takes X^2 to $X \otimes (e \otimes v)$, up to some constant. Then we apply another vector $f \otimes v'$, and obtain, again up to some fixed constants,

$$(e \otimes v')(e \otimes v) + X \otimes \theta(vv') + \omega(v, v')XH.$$

The first component belongs to $S^2(U \otimes V)$, the second one to $\mathfrak{sl}_2 \otimes \mathfrak{h}$, and the third one to $S^2\mathfrak{sl}_2$. The contribution of the first component to $\mathfrak{sl}_2 \otimes \mathfrak{h} \subset S^2(U \otimes V)$ is $e^2 \otimes \theta(vv') = X \otimes \theta(vv')$. We conclude that the Cartan square of $\mathfrak{g}$ does not contain $\mathfrak{sl}_2 \otimes \mathfrak{h} \oplus \mathfrak{sl}_2 \otimes \mathfrak{h} \subset S^2U \otimes S^2V \oplus \mathfrak{sl}_2 \otimes \mathfrak{h}$, but meets it along some diagonal copy of $\mathfrak{sl}_2 \otimes \mathfrak{h}$. This implies our claim. $\square$

3.2. Interpretation of Vogel's parameters

It is now easy to compute the Casimir eigenvalues of our two nontrivial components of $S^2\mathfrak{g}$:

$$\begin{aligned} C_{Y_2(\alpha)} &= (2\alpha_0 + 2\rho, 2\alpha_0) = 2\frac{v+3}{v+2}, \\ C_{Y_2}(\beta) &= (\alpha_0 + \tilde{\alpha} + 2\rho, \alpha_0 + \tilde{\alpha}) = \frac{v+h+2}{v+2}. \end{aligned}$$

Corollary 3.2. *Let $h' = v - h$. Normalize Vogel's parameters for $\mathfrak{g} \neq \mathfrak{g}_2$, such that $\alpha = -2$. Then*

$$\beta = h' + 2, \quad \gamma = h + 2, \quad t = v + 2 = \check{h}.$$

Proof. Vogel's parameters are defined by the fact that, with respect to an invariant quadratic form on $\mathfrak{g}$, the Casimir eigenvalue of $\mathfrak{g}$ is $2t$, the nonzero Casimir eigenvalues of $S^2\mathfrak{g}$ are $2(2t - \alpha), 2(2t - \beta), 2(2t - \gamma)$, and $t = \alpha + \beta + \gamma$. We have been working with the Killing form, for which the Casimir eigenvalue of $\mathfrak{g}$ is 1. Rescaling t to be $v + 2$ and plugging into the formulas for $C_{Y_2(\alpha)}, C_{Y_2(\beta)}$we obtain the result. $\square$

Remark. Note that this does not depend on the fact that $\mathfrak{h}$ is simple. If it is not simple, we can choose the highest root for any simple factor and get a corresponding component of $S^2\mathfrak{g}$, whose Casimir eigenvalue is given as before in terms of the dual Coxeter number of the chosen factor. This implies that we cannot have more than two simple factors, and that when we have two, with dual Coxeter numbers h_1 and h_2, then $v = h_1 + h_2$. This actually happens in type B or D. (Beware that this should be understood up to the symmetry of the Dynkin diagram: in type D_4 we get three different components in $S^2\mathfrak{g}$, but they are permuted by the triality automorphisms and their sum must be considered as simple.)

If $\mathfrak{h}$ is simple, the formula $(\chi + 2\rho_{\mathfrak{h}}, \chi) = \frac{2v+1}{4}(\tilde{\alpha}, \tilde{\alpha})$ gives $c_V = \frac{2v+1}{4h}$, and we get

$$\dim \mathfrak{h} = \frac{c_V}{e_V} = \frac{v(2v+1)}{h' + 2}.$$

In general, Vogel's dimension formula is

$$\dim \mathfrak{g} = d(h, h') = \frac{(h + h' + 3)(2h + h' + 2)(h + 2h' + 2)}{(h + 2)(h' + 2)}.$$

We have the following curious consequence. Parametrize $\mathfrak{g}$ by h and h'. We ask: What values of h and h' can give rise to a $\mathfrak{g}$ such that $\mathfrak{h}$ is simple and V is irreducible? In this case $\mathfrak{h}$ is parametrized by h' and $h - h'$. Thus, $d(h, h') = d(h', h - h') + 3 + 4(h + h')$, which is equivalent to the identity $(h + 1)(h - 2h' + 2) = 0$. Thus, such $\mathfrak{g}$ must be in the symplectic series $h = -1$, or the exceptional series $h = 2h' - 2$!

3.3. Interpretation of h'

Suppose that we are not in type A, so that the adjoint representation is supported on a fundamental weight ω. Let $\alpha_{ad} = \omega_0 - \chi$ denote the corresponding simple root dual to ω. Since the highest root $\tilde{\alpha}$ of $\mathfrak{h}$ is not the highest root of $\mathfrak{g}$, one can find a simple root α such that $\tilde{\alpha} + \alpha$ is again a root, and the only possibility is $\alpha = \alpha_{ad}$. Thus, $\tilde{\alpha} + \alpha_{ad}$ and by symmetry $\psi = \alpha_0 - \tilde{\alpha} - \alpha_{ad}$ both belong to Φ_1. Suppose that $V = V_\chi$ is fundamental, and let α_χ be the corresponding simple root.

Proposition 3.1. *$\phi = \psi - \alpha_{ad} - \alpha_\chi$ is the highest root of $\mathfrak{g}$ orthogonal to α_0 and $\tilde{\alpha}$.*

Proof. We first prove that ϕ is a root. First note that

$$(\psi, \alpha_\chi) = (\chi - \tilde{\alpha}, \alpha_\chi) = (\alpha_\chi, \alpha_\chi)/2 - (\tilde{\alpha}, \alpha_\chi).$$

If we are not in type C, then $(\tilde{\alpha}, \alpha_\chi) = 0$. Indeed, $\tilde{\alpha}$ is a fundamental weight and α_χ is a simple root. So if this were nonzero we would get $\tilde{\alpha} = \chi$, which cannot be since we know that χ is minuscule. Thus, $(\psi, \alpha_\chi) > 0$ and $\psi' = \psi - \alpha_\chi$ is a root. Moreover,

$$(\psi, \alpha_{ad}) = (\alpha_0, \alpha_0)/4 - (\chi, \chi) + (\chi, \tilde{\alpha}) = -(\alpha_0, \alpha_0)/2 + (\tilde{\alpha}, \tilde{\alpha})\chi(H_{\tilde{\alpha}})/2,$$

where $\chi(H_{\tilde{\alpha}})$ is a positive integer (in fact equal to one, since we know that V is minuscule). Thus, $(\psi, \alpha_{ad}) \geqslant 0$ and $(\psi', \alpha_{ad}) \geqslant -(\alpha_\chi, \alpha_{ad}) = (\alpha_\chi, \chi) > 0$. We conclude that $\psi' - \alpha_{ad} = \phi$ is a root.

Since $\phi = 2\chi - \tilde{\alpha} - \alpha_\chi$, it is clearly orthogonal to α_0. Moreover, using again that $(\tilde{\alpha}, \alpha_\chi) = 0$ if we are not in type C, we have $(\phi, \tilde{\alpha}) = (2\chi - \tilde{\alpha}, \tilde{\alpha}) = 0$, since we have just computed that $(\chi, \tilde{\alpha}) = (\tilde{\alpha}, \tilde{\alpha})/2$. To conclude that ϕ is the highest root orthogonal to both α_0 and $\tilde{\alpha}$, we use the following characterization of the highest root of a root system.

Lemma 3.2. *The highest root of an irreducible root system is the only long root γ such that $(\gamma, \alpha) \geqslant 0$ for any simple root α.*

We apply this lemma to $\phi = 2\chi - \tilde{\alpha} - \alpha_\chi = -\tilde{\alpha} - \sum_{\alpha \neq \alpha_\chi} c_{\alpha,\alpha_\chi} \omega_\alpha$, where α belongs to the set of simple roots and c_{α,α_χ} is the corresponding Cartan integer. Since $\alpha \neq \alpha_\chi$, we know that $c_{\alpha,\alpha_\chi} \leqslant 0$, and hence $(\phi, \alpha) \geqslant 0$ for every simple root $\alpha \neq \alpha_{ad}$.

It remains to check whether ϕ is long. Remember that $(\chi, \chi) = \frac{3}{4}(\tilde{\alpha}, \tilde{\alpha})$, $(\tilde{\alpha}, \alpha_\chi) = 0$ and $(\tilde{\alpha}, \chi) = (\tilde{\alpha}, \tilde{\alpha})/2$. We compute that $(\phi, \phi) = 2(\tilde{\alpha}, \tilde{\alpha}) - (\alpha_\chi, \alpha_\chi) \geqslant (\tilde{\alpha}, \tilde{\alpha})$. Therefore, ϕ is long (and we must have equality, so that α_χ is also long). $\square$

Note that, say in the simply laced case,

$$\begin{aligned}(2\rho, \phi) &= (2\rho, \alpha_0) - (2\rho, \tilde{\alpha}) - 3(\alpha_0, \alpha_0) = (v + 1 - h + 1 - 3)(\alpha_0, \alpha_0) \\ &= (h' - 1)(\alpha_0, \alpha_0).\end{aligned}$$

We have therefore isolated three roots $\alpha_0, \tilde{\alpha}, \phi$ of heights $v+1, h-1, h'-1$, respectively.

4. Proof of the main result

4.1. The weights of V and their heights

Our next observation concerns the distribution of the rational numbers $(\rho_{\mathfrak{h}}, \mu)$, when μ describes the set of weights of V. A natural scale for these numbers is the length $(\tilde{\alpha}, \tilde{\alpha})$ of the long roots. We denote by S_p the string of numbers $(p/2 - x)(\tilde{\alpha}, \tilde{\alpha})/2$, for $x = 0, 1, \ldots, p$.

Proposition 4.1. *The values $(\rho_{\mathfrak{h}}, \mu)$, for μ a weight of V, can be arranged into the union of the three strings S_{v-1}, S_{h-1}, $S_{h'-1}$.*

Our main theorem easily follows from this fact: a set of weights μ in V contributing to a string S_p of values of $(\rho_{\mathfrak{h}}, \mu)$ gives a set of roots $\beta = \omega_0 + \mu$ in Φ_1 with

$$\begin{aligned}(\tilde{\alpha}, \beta) &= (\alpha_0, \alpha_0)/2,\\ (\rho, \beta) &= \frac{1+v}{2}(\alpha_0, \alpha_0)/2 + (\rho_{\mathfrak{h}}, \mu) = \left(\frac{1+v}{2} + p/2 - x\right)(\alpha_0, \alpha_0)/2, \quad 0 \leqslant x \leqslant p.\end{aligned}$$

The contribution of this subset of Φ_1 to the Weyl dimension formula is therefore

$$C_p = \prod_{\beta} \frac{(\rho + k\tilde{\alpha}, \beta)}{(\rho, \beta)} = \prod_{x=0}^{p} \frac{\dfrac{1+v+p}{2} - x + k}{\dfrac{1+v+p}{2} - x} = \frac{\binom{\frac{v+p+1}{2}+k}{k}}{\binom{\frac{v-p-1}{2}+k}{k}}.$$

Our proof of the proposition is a case-by-case check. One has to be careful about the case where $h' \leqslant 0$, since the string $S_{\frac{h'-1}{2}}$ is no longer defined. Since for $p > 0$, we have $C_{-p} = C_{p+1}^{-1}$, we should interpret S_{-p} as *suppressing a string* S_{p+1}. We then easily check the rather surprising fact that, interpreted that way, the proposition also holds for $h' < 0$.

4.2. Relation with Knop's construction of simple singularities

Holweck observed that the fact that we can arrange the values of (ρ, β), for $\beta \in \Phi_1$, in no more than three strings, has a curious relation with the work of Knop on simple singularities. Knop proved [11] that if $Y^{\perp} \subset \mathbb{P}\mathfrak{g}$ is a hyperplane Killing orthogonal to a regular nilpotent element $Y \in \mathfrak{g}$, the intersection of this hyperplane with the adjoint variety $X_{ad} \subset \mathbb{P}\mathfrak{g}$ (the projectivization of the minimal nontrivial nilpotent orbit) has an isolated singularity which is simple, of type given by the subdiagram of the Dynkin diagram of $\mathfrak{g}$ obtained from the long simple roots.

We can choose $Y = \sum_{\alpha \in \Delta} X_\alpha$, where Δ denotes the set of simple roots and X_α is a generator of the root space $\mathfrak{g}_\alpha$. The orthogonal hyperplane contains the lowest root space $\mathfrak{g}_{-\tilde{\alpha}} \in X_{ad}$. Let P denote the parabolic subgroup of the adjoint group of $\mathfrak{g}$, which stabilizes $\mathfrak{g}_{\tilde{\alpha}}$, and let U denote its unipotent radical. Being unipotent, U can be identified, through the exponential map, with its algebra $\mathfrak{u}$, a basis of which is given by the root spaces $\mathfrak{g}_\beta$ with $\beta \in \Phi_1 \cup \{\alpha_0\}$. The scalar product with Y defines on the Lie algebra $\mathfrak{u}$ the function $f(X) = K(Y, exp(X)X_{\tilde{\alpha}})$. The quadratic part of this function is

$$q(X, X') = \tfrac{1}{2}K(Y, ad(X')ad(X)X_{\tilde{\alpha}}) = \tfrac{1}{2}K(ad(X)Y, ad(X')X_{\tilde{\alpha}}).$$

The kernel of this quadratic form thus contains the kernel of the map $X \mapsto [Y, X]$, $X \in \mathfrak{u}$.

Suppose for simplicity that $\mathfrak{g}$ is simply laced. Let

$$\mathfrak{g}_l = \bigoplus_{\gamma\in\Phi_1\cup\{\alpha_0\},\ (\rho,\gamma)=l} \mathfrak{g}_\gamma.$$

Then $ad(Y)$ maps $\mathfrak{g}_l$ to $\mathfrak{g}_{l+(\tilde{\alpha},\tilde{\alpha})/2}$. In particular, the kernel of $ad(Y)_{|\mathfrak{g}_l}$ has dimension at least $\dim \mathfrak{g}_l - \dim \mathfrak{g}_{l+(\tilde{\alpha},\tilde{\alpha})/2}$. Since $\mathfrak{g}_v = \mathfrak{g}_{\alpha_0}$ is one-dimensional, we deduce that for all l,

$$\dim \mathfrak{g}_l \leqslant 1 + \operatorname{corank}(q).$$

The maximal dimension of $\mathfrak{g}_l$ is the minimal number of strings we need to arrange the values of (ρ,β) for $\beta \in \Phi_1$. This number is bounded by three because, since f defines a simple singularity, the corank of its quadratic part must be at most two.

Note that in Knop's work there is no direct proof of this fact. It follows from a numerical criterion and a trick attributed to Saito [11, Lemma 1.5].

5. Gradings

The highest roots α_0 and $\tilde{\alpha}$ both induce 5-gradings on $\mathfrak{g}$. Being orthogonal, they induce a double grading

$$\mathfrak{g}_{ij} = \{X \in \mathfrak{g}, [H_{\alpha_0}, X] = iX, [H_{\tilde{\alpha}}, X] = jX\}.$$

Proposition 5.1. *With the normalization $t = \check{h}$, for $\mathfrak{g}$ of rank at least three, the dimensions of the components of this double grading are given by the following diamond*:

$$\begin{array}{ccccc} & & 1 & & \\ & \beta & 2\gamma-8 & \beta & \\ 1 & 2\gamma-8 & * & 2\gamma-8 & 1 \\ & \beta & 2\gamma-8 & \beta & \\ & & 1 & & \end{array}$$

Proof. Let g_{ij} denote the dimension of $\mathfrak{g}_{ij}$. Since the dual Coxeter number of $\mathfrak{g}$ is t, the dimension of the positive part of the 5-grading of $\mathfrak{g}$ is $2g_{11} + g_{01} + 1 = 2t - 3 = 2\beta + 2\gamma - 7$. Since the dual Coxeter number of $\mathfrak{h}$ is h, the dimension of the positive part of the 5-grading of $\mathfrak{h}$ is $g_{01} + 1 = 2h - 3 = 2\gamma - 7$. Hence the claim. $\square$

Corollary 5.2. *With the normalization $t = \check{h}$, the integer β is the number of roots θ in Φ_1 such that $\tilde{\alpha} + \theta$ is still a root.*

Proof. Let $\theta \in \Phi_1$ be such that $\mathfrak{g}_\theta \subset \mathfrak{g}_{11}$. This means that $\theta(H_{\tilde{\alpha}}) = 1$. Then $\theta^* = \alpha_0 - \theta$ is also a root, and $\theta^*(H_{\tilde{\alpha}}) = -1$; thus $s_{\tilde{\alpha}}(\theta^*) = \tilde{\alpha} + \theta^*$ is a root. Conversely, if $\tilde{\alpha} + \theta^*$

is a root, $(\tilde{\alpha}+\theta^*)(H_{\tilde{\alpha}}) = 2+\theta^*(H_{\tilde{\alpha}})$, as well as $\theta^*(H_{\tilde{\alpha}})$, belongs to $\{-2,-1,0,1,2\}$, and hence $\theta^*(H_{\tilde{\alpha}}) = -1$ and we can recover $\theta \in \Phi_1$. The $\mathfrak{g}_2$ case may be verified directly. $\square$

Let $\mathfrak{g}_{00}^* \subset \mathfrak{g}_{00}$ denote the common centralizer of X_{α_0} and $X_{\tilde{\alpha}}$. We have $\mathfrak{g}_{00} = \mathfrak{g}_{00}^* \oplus \mathbb{C}H_{\alpha_0} \oplus \mathbb{C}H_{\tilde{\alpha}}$.

Proposition 5.3. $\mathfrak{g}_{11}$ is endowed with a $\mathfrak{g}_{00}^*$-invariant nondegenerate quadratic form.

Proof. For $Y, Z \in \mathfrak{g}_{11}$, let

$$Q(Y,Z) = K([X_{\tilde{\alpha}}, Y], [X_{-\alpha_0}, Z]).$$

This bilinear form is obviously $\mathfrak{g}_{00}^*$-invariant. We check whether it is symmetric:

$$\begin{aligned} Q(Y,Z) &= K(X_{\tilde{\alpha}}, [Y, [X_{-\alpha_0}, Z]]) \\ &= K(X_{\tilde{\alpha}}, [Z, [X_{-\alpha_0}, Y]]) + K(X_{\tilde{\alpha}}, [X_{-\alpha_0}, [Y, Z]]) \\ &= Q(Z,Y) + \Omega(Y,Z)K(X_{\tilde{\alpha}}, H_{\alpha_0}) \\ &= Q(Z,Y). \end{aligned}$$

Recall that $K(\mathfrak{g}_\beta, \mathfrak{g}_\gamma) \neq 0$ if and only if $\beta+\gamma = 0$. To prove that Q is nondegenerate, we must therefore check whether for each root space $\mathfrak{g}_\theta$ in $\mathfrak{g}_{11}$, $\tilde{\alpha}+\theta$ is a root—this follows from the corollary above. $\alpha_0 - (\tilde{\alpha}+\theta)$ is also a root—this follows from the fact that $\tilde{\alpha}+\theta$ is in Φ_1. $\square$

We thus get an invariant map $\mathfrak{g}_{00}^* \to \mathfrak{so}_\beta$, which turns out to be *surjective*. We can thus write $\mathfrak{g}_{00} = \mathbb{C}^2 \times \mathfrak{so}_\beta \times \mathfrak{k}$ for some reductive subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Our double grading of $\mathfrak{g}$ takes the form

$$\begin{array}{ccccccc} & & & \mathbb{C} & & & \\ & \mathbb{C}^\beta & & U^{2\gamma-8} & & \mathbb{C}^\beta & \\ \mathbb{C} & U^{2\gamma-8} & \mathbb{C}^2 & \times\ \mathfrak{so}_\beta\ \times & \mathfrak{k} & U^{2\gamma-8} & \mathbb{C} \\ & \mathbb{C}^\beta & & U^{2\gamma-8} & & \mathbb{C}^\beta & \\ & & & \mathbb{C} & & & \end{array}$$

Note that U is a symplectic $\mathfrak{k}$-module. Now, consider the 5-step simple grading that we obtain by taking diagonals. Since $\mathfrak{so}_\beta \oplus \mathbb{C}^\beta \oplus \mathbb{C}^\beta \oplus \mathbb{C} = \mathfrak{so}_{\beta+2}$, we get

$$\mathfrak{g} = \mathbb{C}^{\beta+2} \oplus V^{4\gamma-16} \oplus (\mathbb{C} \times \mathfrak{so}_{\beta+2} \times \mathfrak{k}) \oplus V^{4\gamma-16} \oplus \mathbb{C}^{\beta+2}.$$

Remarkably, this induces a very simple $\mathbb{Z}_2$-grading

$$\mathfrak{g} = (\mathfrak{so}_{\beta+4} \times \mathfrak{k}) \oplus W^{8\gamma-32}.$$

The subalgebra $\mathfrak{k}$ and the module $W^{8\gamma-32}$ are given by the following table:

β	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{so}_{\beta+4}$	W
1	$\mathfrak{sp}_{2n}$	$\mathfrak{sp}_{2n-4}$	$\mathfrak{so}_5 = \mathfrak{sp}_4$	$A_4 \otimes B_{2n-4}$
2	$\mathfrak{sl}_n$	$\mathfrak{gl}_{n-4}$	$\mathfrak{so}_6 = \mathfrak{sl}_4$	$A_4^* \otimes B_{n-4} \oplus A_4 \otimes B_{n-4}^*$
4	$\mathfrak{so}_n$	$\mathfrak{so}_{n-8}$	$\mathfrak{so}_8$	$A_8 \otimes B_{n-8}$
5	$\mathfrak{f}_4$		$\mathfrak{so}_9$	Δ
6	$\mathfrak{e}_6$	$\mathbb{C}$	$\mathfrak{so}_{10}$	$\Delta_+ \oplus \Delta_-$
8	$\mathfrak{e}_7$	$\mathfrak{sl}_2$	$\mathfrak{so}_{12}$	$A_2 \otimes \Delta_+$
12	$\mathfrak{e}_8$		$\mathfrak{so}_{16}$	Δ_+
a	$F3G(a, r)$	$F3G(a, r-2)$	$\mathfrak{so}_{a+4}$	$\mathbb{C}^{8a(r-2)}$

5.1. More dimension formulas

Recall from Proposition 3.1 that $Y_2(\beta)$ has the highest weight $\tilde{\alpha} + \alpha_0$. Therefore, the diamond of Proposition 5.1 also gives the number of roots having a given scalar product with the highest weights of $\mathfrak{g}$ and $Y_2(\beta)$. From Corollary 4.1, we know the values of (ρ, α) when α describes the roots in Φ_1. Among these, the β positive roots from $\mathfrak{g}_{1,-1}$ (and $\mathfrak{g}_{1,1}$, symmetrically) contribute a string of length $\beta - 1$, the middle point having multiplicity two (just like the weights of the natural representation of $\mathfrak{so}_\beta$, in accordance with Proposition 5.3). Finally, since $\mathfrak{g}_{0,1}$ is part of the 5-step adjoint grading of $\mathfrak{h}$, its contribution can be described by Proposition 4.1 with the pair h, h' changed into $h', h - h'$. Using the Weyl dimension formula, we get

Theorem 5.4. *The dimensions of the Cartan products of powers of* $\mathfrak{g}$ *and* $Y_2(\beta)$ *are given by the universal formula*

$$\dim \mathfrak{g}^{(k)} Y_2{}^{(l)} = F(\beta, \gamma, k, l) A(\beta, \gamma, k+l) B(\beta, \gamma, l) C(\beta, \gamma, k+2l) C(\beta, \gamma, k-\gamma+3)$$

with

$$F(\beta, \gamma, k, l) = \frac{(\beta+\gamma-3+2k+2l)(\gamma-3+2l)(\beta/2+\gamma-3+k+2l)(\beta/2+k)}{(\beta+\gamma-3)(\gamma-3)(\beta/2+\gamma-3)\beta/2},$$

$$A(\beta, \gamma, k) = \frac{\binom{\beta+\gamma/2-3+k}{k}\binom{\gamma+\beta/2-4+k}{k}\binom{\gamma-3+k}{k}}{\binom{\beta/2+k}{k}\binom{-1+\beta+k}{k}\binom{-1+\gamma/2+k}{k}},$$

$$B(\beta,\gamma,k) = \frac{\binom{\gamma-\beta/2-3+k}{k}\binom{\gamma/2+\beta/2-3+k}{k}\binom{\gamma-4+k}{k}}{\binom{\beta/2-1+k}{k}\binom{\gamma/2-\beta/2-1+k}{k}},$$

$$C(\beta,\gamma,k) = \frac{\binom{\beta+\gamma-4+k}{k}}{\binom{\gamma-3+k}{k}}.$$

6. The modules $Y_k(\beta)$ and $Y_k(\gamma)$

For each simple Lie algebra $\mathfrak{g}$ we have obtained a general formula for the dimension of its kth Cartan power as a rational function of α, β, γ, symmetric with respect to β and γ. Following Vogel, the three numbers should play a completely symmetric role, and by permutation we should get the dimensions of (virtual) $\mathfrak{g}$-modules $Y_k(\beta)$ and $Y_k(\gamma)$. We first check whether this is indeed the case. The formula predicts that these modules must be zero when k becomes large, but an interesting pattern shows up in the classical cases.

6.1. *Identification*

The formula for the dimension of $Y_k(\beta)$ is

$$\dim Y_k(\beta) = \frac{2\gamma-(2k-3)\beta-4}{2\gamma-(k-3)\beta-4}\prod_{i=1}^{k} \times\frac{(2\gamma-(i-3)\beta-2)(2\gamma-(i-3)\beta-4)((\gamma-(i-3)\beta-4)}{i\beta((i-1)\beta+2)(\gamma-(i-1)\beta)}.$$

When k is small enough, $Y_k(\beta)$ is an irreducible module whose highest weight is given by Proposition 6.1 below. But the formula above may give a nonzero integer when k is too big for the hypothesis of this proposition to hold. We check case by case whether, nevertheless, this integer is still the dimension of an irreducible module, or possibly *the opposite* of the dimension of an irreducible module. This means that $Y_k(\beta)$ should be interpreted as a virtual module, which is a true module for small k, possibly the opposite of a module for intermediate values of k, and zero for k sufficiently large. In the second situation, we put a minus sign before the highest weight of the corresponding module in the lists below.

a. $Y_k(\beta)$ for $\mathfrak{sp}_{2l}$ (note that we have the fold of $\mathfrak{sl}_{2l+2}$):

k	0	1	2	...	l	$l+1, l+2$	$l+3$	...	$2l+2$	$2l+3$	$\geqslant 2l+4$
$Y_k(\beta)$	$\mathbb{C}$	$2\omega_1$	$2\omega_2$	$\cdots$	$2\omega_l$	0	$-2\omega_l$	$\cdots$	$-2\omega_1$	$\mathbb{C}$	0

b. $Y_k(\beta)$ for $\mathfrak{sl}_{l+1}$, for $l=2m-1$ odd and $l=2m$ even, respectively:

k	0	1	$\cdots$	$m-1$	m				
$Y_k(\beta)$	$\mathbb{C}$	$\omega_1+\omega_{2m-1}$	$\cdots$	$\omega_{m-1}+\omega_{m+1}$	$2\omega_m$				
				$m+1$	$m+2$	$\cdots$	$2m+1$	$2m+2$	$\geqslant 2m+3$
				$-2\omega_m$	$-(\omega_{m-1}+\omega_{m+1})$	$\cdots$	$-(\omega_1+\omega_m)$	$-\mathbb{C}$	0

k	0	1	$\cdots$	m	$m+1$				
$Y_k(\beta)$	$\mathbb{C}$	$\omega_1+\omega_{2m}$	$\cdots$	$\omega_m+\omega_{m+1}$	0				
					$m+2$	$\cdots$	$2m+1$	$2m+2$	$\geqslant 2m+3$
					$-(\omega_m+\omega_{m+1})$	$\cdots$	$-(\omega_1+\omega_{2m})$	$-\mathbb{C}$	0

c. $Y_k(\beta)$ for $\mathfrak{so}_{2l+1}$, for $l=2m-1$ odd and $l=2m$ even, respectively:

k	0	1	$\cdots$	$m-1$	m	$m+1$	$\cdots$	$l=2m-1$	$\geqslant l+1$
$Y_k(\beta)$	$\mathbb{C}$	ω_2	$\cdots$	ω_{2m-2}	$2\omega_{2m-1}$	ω_{2m-3}	$\cdots$	ω_1	0

k	0	1	$\cdots$	$m-1$	m	$m+1$	$\cdots$	$l=2m$	$\geqslant l+1$
$Y_k(\beta)$	$\mathbb{C}$	ω_2	$\cdots$	ω_{2m-2}	$2\omega_{2m}$	ω_{2m-1}	$\cdots$	ω_1	0

d. $Y_k(\beta)$ for $\mathfrak{so}_{2l}$, $l\geqslant 4$, for $l=2m-1$ odd and $l=2m$ even, respectively:

k	0	1	$\cdots$	$m-2$	$m-1,m$	$m+1$	$\cdots$	$2m-1$	$\geqslant 2m$
$Y_k(\beta)$	$\mathbb{C}$	ω_2	$\cdots$	ω_{2m-4}	$\omega_{2m-2}+\omega_{2m-1}$	ω_{2m-3}	$\cdots$	ω_1	0

k	0	1	$\cdots$	$m-1$	m	$m+1$	$\cdots$	$2m$	$\geqslant l+1$
$Y_k(\beta)$	$\mathbb{C}$	ω_2	$\cdots$	ω_{2m-2}	0	ω_{2m-2}	$\cdots$	$\mathbb{C}$	0

e. $Y_k(\beta)$ for the exceptional Lie algebras:

	k	0	1	2	3	4	5	6	$\geqslant 7$
$\mathfrak{g}_2$	$Y_k(\beta)$	$\mathbb{C}$	ω_2	$2\omega_1$	ω_1	0	0	0	0
$\mathfrak{f}_4$	$Y_k(\beta)$	$\mathbb{C}$	ω_1	$2\omega_4$	ω_3	ω_4	0	0	0
$\mathfrak{e}_6$	$Y_k(\beta)$	$\mathbb{C}$	ω_2	$\omega_1+\omega_6$	$\omega_1+\omega_6$	ω_2	$\mathbb{C}$	0	0
$\mathfrak{e}_7$	$Y_k(\beta)$	$\mathbb{C}$	ω_1	ω_6	$2\omega_7$	0	0	0	0
$\mathfrak{e}_8$	$Y_k(\beta)$	$\mathbb{C}$	ω_8	ω_1	0	$-\omega_1$	$-\omega_8$	$-\mathbb{C}$	0

f. $Y_k(\gamma)$:

	k	2	3	4	$\geqslant 5$
$\mathfrak{sp}_{2l}$	$Y_k(\gamma)$	ω_2	0	0	0
$\mathfrak{sl}_l$	$Y_k(\gamma)$	$\omega_1+\omega_{l-1}$	$\mathbb{C}$	0	0
$\mathfrak{so}_m$	$Y_k(\gamma)$	$2\omega_1$	0	0	0
$\mathfrak{e},\mathfrak{f},\mathfrak{g}$	$Y_k(\gamma)$	0	$-\mathfrak{g}$	$-\mathbb{C}$	0

Note that the $Y_k(\beta)$'s travel nicely along the Dynkin diagram in a wave (that gets reflected when it hits the end of a diagram or collides with something else in the diagram, becoming negative if there is no arrow).

6.2. Gradings and the $Y_k(\beta)$'s

Let $\beta_1 = \alpha_0$ denote the highest root, let $\beta_2 = \tilde{\alpha}$ denote the highest long root orthogonal to α_0, β_3 the highest long root orthogonal to α_0 and β_2, etc.

Proposition 6.1. *If $\sigma_i = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_i$ is dominant for $i \leqslant k$, then σ_k is the highest weight of an irreducible component of $S^k\mathfrak{g}$.*

This module turns out to be $Y_k(\beta)$, the module that we identified from its dimension. Of course this does not explain what happens when $Y_k(\beta)$ is only a virtual module, and in fact there are also cases for which $Y_k(\beta)$ is an actual module whose presence is not accounted for by the proposition. To be precise, this happens when $k \geqslant m$ for $\mathfrak{so}_{4m-3}$, $\mathfrak{so}_{4m-2}$, $\mathfrak{so}_{4m-1}$ or $\mathfrak{so}_{4m}$, when $k \geqslant 2$ for $\mathfrak{g}_2$, when $k \geqslant 3$ for $\mathfrak{f}_4$ and $\mathfrak{e}_6$.

Note that the dominance condition in the hypothesis of Proposition 6.1 is not automatic, and will be essential in the following construction. We associate to the k roots $\beta_1, \ldots, \beta_k$ a $\mathbb{Z}^k$-grading of $\mathfrak{g}$,

$$\mathfrak{g}_{l_1 \cdots l_k} = \{X \in \mathfrak{g}, [H_{\beta_i}, X] = l_i X, i = 1 \ldots k\}.$$

Lemma 6.2. *Suppose that $l_1, \ldots, l_k \geqslant 0$ and $\mathfrak{g}_{l_1 \cdots l_k} \neq 0$. Then $l_1 + \cdots + l_k \leqslant 2$.*

Note that for a given $\mathfrak{g}$ which is not $\mathfrak{so}_n$ for some $n \geqslant 5$, there is no ambiguity in defining the integer β. This implies that for any k as above, the components $\mathfrak{g}_{0..1..1..0}$ of our k-dimensional grading have the same dimension, β, by Proposition 5.1; in particular they are nonzero.

If $\mathfrak{g} = \mathfrak{so}_n$ for some $n \geqslant 5$, the Lie subalgebra we denoted $\mathfrak{h}$ is the product of $\mathfrak{sl}_2$ and $\mathfrak{so}_{n-4}$, and we can choose for β_2 the highest root of either algebra. If we choose that of $\mathfrak{sl}_2$, we cannot go further: $\beta_1 + \beta_2 + \beta_3$ will not be dominant. If we choose the highest root of $\mathfrak{so}_{n-4}$, we can go further, but there is no more choice, we can only take $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ and again the components $\mathfrak{g}_{0..1..1..0}$ of the grading have the same dimension, four.

Example. For $\mathfrak{g} = \mathfrak{e}_7$, the highest root is $\alpha_0 = \beta_1 = \omega_1$. The highest root orthogonal to α_0 is the highest root of a subsystem of type D_6. We get $\beta_2 = \omega_6 - \omega_1$ and $\beta_1 + \beta_2 = \omega_6$ is dominant. For the next step, the roots orthogonal both to β_1 and β_2, i.e., both to ω_1 and ω_6, form a reducible subsystem of type $D_4 \times A_1$, and we have two candidates for the next highest root. If we choose $\beta_3 = \omega_4 - \omega_1 - \omega_6$, the highest root of the D_4 part, then $\beta_1 + \beta_2 + \beta_3$ is not dominant. The only possible choice is therefore $\beta_3 = \alpha_7 = 2\omega_7 - \omega_6$, for which $\beta_1 + \beta_2 + \beta_3 = 2\omega_7$ is dominant. Then the process stops.

We obtain a three-dimensional grading of $\mathfrak{e}_7$ with three types of nonzero components: the six components $\mathfrak{g}_{\pm 200}, \mathfrak{g}_{0\pm 20}, \mathfrak{g}_{00\pm 2}$ are one-dimensional; the 12 components $\mathfrak{g}_{\pm 1\pm 10}, \mathfrak{g}_{\pm 10\pm 1}, \mathfrak{g}_{0\pm 1\pm 1}$ have dimension eight and must be interpreted as copies of the (complexified) octonions; the central component $\mathfrak{g}_{000} = \mathbb{C}^3 \oplus \mathfrak{so}_8$. This is very close to the triality construction of $\mathfrak{e}_7$ as $\mathfrak{g}(\mathbb{O}, \mathbb{H})$ [13].

Corollary 6.3. *For each* $1 \leqslant i < k$, *we have* $(2\rho, \beta_i - \beta_{i+1}) = \beta(\alpha_0, \alpha_0)$.

Proof. Recall that 2ρ is the sum of the positive roots. If γ is such a positive root, then $\mathfrak{g}_\gamma$ is contained in one of the $\mathfrak{g}_{l_1 \dots l_k}$, and the fact that $\beta_1 + \dots + \beta_j$ is dominant for all j implies that $l_1 + \dots + l_j \geqslant 0$ for all j. Conversely, such a component $\mathfrak{g}_{l_1 \dots l_k}$ is the sum of positive root spaces, except of course the central component $\mathfrak{g}_{0 \dots 0}$.

The integer $\gamma(H_{\beta_i})$ is equal to 2 if $\mathfrak{g}_\gamma = \mathfrak{g}_{0 \dots 2 \dots 0}$ with the 2 in position i, 1 if $\mathfrak{g}_\gamma \subset \mathfrak{g}_{0..1..1..0}$ or $\mathfrak{g}_{0..1..-1..0}$, both of dimension β, 1 again for $\mathfrak{g}_{0..1..0}$, of dimension with the 1 in position i, and zero otherwise. Since $\mathfrak{g}_{0..1..0}$ has dimension $2\gamma - 8 - 2(k-2)\beta$, we conclude that

$$2\rho(H_{\beta_i}) = (k - 1 + k - i)\beta + 2\gamma - 8 - 2(k-2)\beta + 2 = 2\gamma + (3 - i)\beta - 6,$$

and the claim follows. $\square$

Corollary 6.4. *The Casimir eigenvalue of* $Y_k(\beta)$ *is* $2kt - k(k-1)\beta$.

Proof of the Lemma. Let θ be some root such that $\mathfrak{g}_\theta \subset \mathfrak{g}_{l_1 \cdots l_k}$. If some l_i equals 2, then θ must equal β_i and the other coefficients vanish. So we suppose that $l_{i_1} = \cdots = l_{i_p} = 1$, and the other coefficients are zero. Using the orthogonality of β_i's, we can write

$$\theta = \frac{1}{2}(\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p} + \gamma),$$

where γ is orthogonal to β_i's. Suppose that i_1 is smaller than the other i_q's and apply the symmetry $s = s_{i_2} \cdots s_{i_p}$. We conclude that

$$s(\theta) = \frac{1}{2}(\beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p} + \gamma)$$

is again a root. Since $(s(\theta), \beta_1 + \cdots + \beta_{i_1}) > 0$, it must be a positive root. But

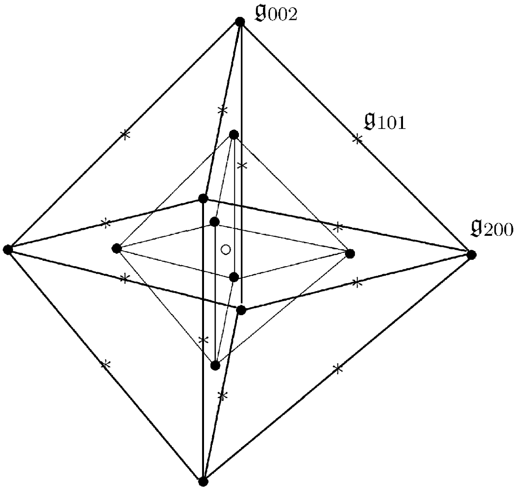
$$(\beta_1 + \cdots + \beta_k, s(\theta)) = 1 - (p-1) = 2 - p.$$

Since $\beta_1 + \cdots + \beta_k$ is supposed to be dominant, this must be a nonnegative integer. Hence $p \geqslant 2$, which is what we wanted to prove. $\square$

Proof of Proposition 6.1. The case $k > 3$ only happens for the classical Lie algebras, for which we can exhibit the highest weight vector of weight σ_k as a determinant, or a Pfaffian, in terms of the basis of the natural representation preserved by the maximal torus:

$$\begin{array}{ll} \mathfrak{sl}_n & \det(e_p \otimes e^*_{n+1-q})_{1 \leqslant p,q \leqslant k}, \\ \mathfrak{sp}_{2n} & \det(e_p \circ e_q)_{1 \leqslant p,q \leqslant k}, \\ \mathfrak{so}_n & \mathrm{Pf}(e_p \wedge e_q)_{1 \leqslant p,q \leqslant k}. \end{array}$$

So we focus on the case $k = 3$. We have a $\mathbb{Z}^3$-grading of $\mathfrak{g}$ with 6 terms of dimension one, 12 of dimension β, 6 of dimension $2\gamma - 2\beta - 8$, and the central term $\mathfrak{g}_{000}$. The six terms of type $\mathfrak{g}_{200}$ can be represented as the vertices of a square pyramid; then the 12 terms of type $\mathfrak{g}_{110}$ are the middle points of the edges.



We have chosen vectors $X_{\pm\beta_i}$ generating three commuting $\mathfrak{sl}_2$-triples. We next choose generators X_α for the roots $\alpha \in \Phi_{110}$ such that the corresponding root space $\mathfrak{g}_\alpha \subset \mathfrak{g}_{110}$. We then take bases of $\mathfrak{g}_{-110}$, $\mathfrak{g}_{1-10}$ and $\mathfrak{g}_{-1-10}$ by letting

$$X_{\alpha-\beta_1} = [X_{-\beta_1}, X_\alpha], \quad X_{\alpha-\beta_2} = [X_{-\beta_1}, X_\alpha], \quad X_{\alpha-\beta_1-\beta_2} = [X_{-\beta_1}, X_{\alpha-\beta_2}].$$

An easy consequence of the Jacobi identity is that $[X_{\beta_1}, X_{\alpha-\beta_1}] = X_\alpha$. Note that $\alpha - \beta_1 - \beta_2 = s_{\beta_1}s_{\beta_2}(\alpha)$ is a root, and that its opposite

$$\alpha' = \beta_1 + \beta_2 - \alpha$$

corresponds to another root space in $\mathfrak{g}_{110}$. (We will use the notation $\theta' = \beta_1 + \beta_2 - \theta$ repeatedly in what follows.) Then $[X_\alpha, X_{\alpha'-\beta_2}]$, which is equal to $[X_{\alpha'}, X_{\alpha-\beta_2}]$ by the Jacobi identity, is a nonzero multiple of X_{β_1}. We normalize our root vectors from $\mathfrak{g}_{110}$ so that this multiple is in fact X_{β_1} itself. This means that for all $\alpha \in \Phi_{110}$,

$$K(X_{\beta_1}, X_{-\beta_1}) = K([X_\alpha, X_{\alpha'-\beta_2}], X_{-\beta_1}) = K(X_\alpha, [X_{\alpha'-\beta_2}, X_{-\beta_1}]) = -K(X_\alpha, X_{-\alpha}).$$

We use the same normalization for $\mathfrak{g}_{101}$ and $\mathfrak{g}_{011}$. Note that $2K(X_{\beta_1}, X_{-\beta_1}) = K(H_{\beta_1}, H_{\beta_1}) = 2/(\beta_1, \beta_1)$, twice the inverse of the square length of a long root. In particular, $K(X_\alpha, X_{-\alpha})$ does not depend on $\alpha \in \Phi_{\pm1\pm10}$, $\Phi_{\pm10\pm1}$ or $\Phi_{0\pm1\pm1}$.

Now we introduce the symmetric tensor

$$S_{12} = \sum_{\alpha\in\Phi_{110}} X_\alpha X_{\alpha'}.$$

Here and in what follows, if $X, Y \in \mathfrak{g}$ or a symmetric power of $\mathfrak{g}$, XY will denote the symmetric product $X \circ Y$.

Lemma 6.5. *For $Y, Z \in \mathfrak{g}_{-1-10}$, the bilinear forms*

$$\sum_{\alpha\in\Phi_{110}} K(X_\alpha, Y)K(X_{\alpha'}, Z) \quad \text{and} \quad K([X_{\beta_1}, Y], [X_{\beta_2}, Z])$$

are multiples of one another.

In particular, S_{12} is invariant under the common centralizer of β_1 and β_2.

Proof. Let $Y = X_{-\beta'}$ and $Z = X_{-\gamma'}$ for some roots $\beta, \gamma \in \Phi_{110}$. Then $[X_{\beta_1}, Y] = X_{\beta-\beta_2}$ and $[X_{\beta_2}, Z] = X_{\gamma-\beta_1}$, and hence

$$K([X_{\beta_1}, Y], [X_{\beta_2}, Z]) = K(X_{\beta-\beta_2}, X_{\gamma-\beta_1}) = -K(X_{-\beta'}, X_\gamma) = \delta_{\beta',\gamma} K(X_{-\beta_1}, X_{\beta_1}).$$

Since $\sum_{\alpha\in\Phi_{110}} K(X_\alpha, Y)K(X_{\alpha'}, Z) = \delta_{\beta',\gamma} K(X_{-\beta_1}, X_{\beta_1})^2$, the claim follows. $\square$

We deduce a different proof of Proposition 3.1. We must prove that $S^2\mathfrak{g}$ contains a tensor of weight $\alpha_0 + \tilde{\alpha} = \beta_1 + \beta_2$ which is the highest weight vector, i.e., which is annihilated by any positive root vector.

Corollary 6.6. *The tensor $\Sigma = X_{\beta_1} X_{\beta_2} - \frac{1}{2} S_{12} \in S^2\mathfrak{g}$ is the highest weight vector of weight $\beta_1 + \beta_2$.*

Proof. We must prove that Σ is annihilated by any positive root vector. Since β_1 and $\beta_1 + \beta_2$ are both dominant, a positive root must belong either to Φ_{00}, Φ_{1-1} or Φ_{pq}

with $p, q \geqslant 0$ and $p+q > 0$. Since $\mathfrak{g}_{1+p,1+q} = 0$, our assertion is clear for the latter case. For the first case, it follows from the previous lemma.

It remains to be proven that $ad(X_{\theta-\beta_2})\Sigma = 0$ for $\theta \in \Phi_{11}$. We use the structure constants $N_{\mu,\nu}$ such that $[X_\mu, X_\nu] = N_{\mu,\nu}X_{\mu+\nu}$. Note that $ad(X_{\theta-\beta_2})X_{\beta_1}X_{\beta_2} = X_{\beta_1}X_\theta$.

On the other hand,

$$ad(X_{\theta-\beta_2})S_{12} = 2\sum_{\alpha\in\Phi_{11}} N_{\theta-\beta_2,\alpha}X_{\alpha+\theta-\beta_2}X_{\alpha'}.$$

But $\alpha+\theta-\beta_2$ belongs to Φ_{200}, and hence must be equal to β_1, which forces $\alpha = \theta'$. Our normalization is $N_{\theta-\beta-2,\theta'} = -1$, thus $ad(X_{\theta-\beta_2})S_{12} = 2X_{\beta_1}X_\theta$. This concludes the proof. $\square$

Now we define a tensor $T \in \mathfrak{g}_{110}\otimes\mathfrak{g}_{101}\otimes\mathfrak{g}_{011} \subset S^3\mathfrak{g}$, with the help of which we will construct the highest weight vector of weight $\beta_1+\beta_2+\beta_3$:

$$T = \sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\gamma\in\Phi_{110}\\ \alpha+\beta+\gamma=\beta_1+\beta_2+\beta_3}} N_{\beta_3-\alpha,\beta_1-\beta}X_\alpha X_\beta X_\gamma. \tag{1}$$

We will need the following properties of the structure constants.

Lemma 6.7. *If $\mu \in \Phi_{1-10}$ and $\nu \in \Phi_{01-1}$, then*

$$N_{\mu,\nu} = -N_{\mu+\beta_2,\nu-\beta_2}, \tag{2}$$

$$N_{\mu,\nu} = N_{\mu-\beta_1,\nu}, \tag{3}$$

$$N_{\mu,\nu} = N_{\nu,-\mu-\nu}. \tag{4}$$

Of course, we have similar identities when we permute the indices, e.g., $N_{\mu,\nu} = -N_{\mu+\beta_3,\nu-\beta_3}$ if $\mu \in \Phi_{10-1}$ and $\nu \in \Phi_{0-11}$.

Proof. By definition, $X_\mu = [X_{-\beta_2}, X_{\mu+\beta_2}]$. In the Jacobi identity

$$[[X_{-\beta_2}, X_{\mu+\beta_2}], X_\nu] + [[X_{\mu+\beta_2}, X_\nu], X_{-\beta_2}] + [[X_\nu, X_{-\beta_2}], X_{\mu+\beta_2}] = 0,$$

the first bracket of the second term is in $\mathfrak{g}_{12-1}$, and hence equal to zero. Since $[X_\nu, X_{-\beta_2}] = -X_{\nu-\beta_2}$, the first identity follows. To prove the second one, we use the Jacobi identity

$$[[X_{-\beta_1}, X_\mu], X_\nu] + [[X_\mu, X_\nu], X_{-\beta_1}] + [[X_\nu, X_{-\beta_1}], X_{-\mu}] = 0.$$

Here the first bracket of the last term is in $\mathfrak{g}_{-11-1}$, and hence equal to zero, and we deduce the second identity. Finally, the invariance of the Killing form gives

$$\begin{aligned} N_{\mu,v}K(X_{\mu+v}, X_{-\mu-v}) &= K([X_\mu, X_v], X_{-\mu-v}) = K(X_\mu, [X_v, X_{-\mu-v}]) \\ &= N_{v,-\mu-v}K(X_\mu, X_\mu). \end{aligned}$$

But in the normalization we use, we have seen that $K(X_{\mu+v}, X_{-\mu-v}) = K(X_\mu, X_\mu)$, and the third identity follows. $\square$

Proposition 6.8. *The tensor $\Theta \in S^3\mathfrak{g}$ defined as*

$$\Theta = X_{\beta_1}X_{\beta_2}X_{\beta_3} - X_{\beta_1}S_{23} - X_{\beta_2}S_{13} - X_{\beta_3}S_{12} + T,$$

is the highest weight vector of weight $\beta_1 + \beta_2 + \beta_3$.

Proof. We must check that Θ is annihilated by any positive root vector X_μ. If $\mu(H_{\beta_i})\geqslant 0$ for $i = 1, 2, 3$, and at least one is positive, this is clear since X_μ annihilates every space of type $\mathfrak{g}_{200}$ or $\mathfrak{g}_{110}$. If these three integers vanish, that is, $X_\mu \in \mathfrak{g}_{000}$, this follows from the fact that for $X \in \mathfrak{g}_{0-1-1}$, $Y \in \mathfrak{g}_{-10-1}$ and $Z \in \mathfrak{g}_{-1-10}$,

$$\begin{aligned} &\sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\gamma\in\Phi_{110}\\ \alpha+\beta+\gamma=\beta_1+\beta_2+\beta_3}} N_{\alpha-\beta_3,\beta-\beta_1} K(X_\alpha, X)K(X_\beta, Y)K(X_\gamma, Z) \\ &\quad = K([[X_{\beta_1}, Y], [X_{\beta_2}, Z]], [X_{\beta_3}, X]), \end{aligned}$$

which shows that T must be annihilated by any vector commuting with X_{β_1}, X_{β_2} and X_{β_3}—and X_μ has this property.

Now, since μ is positive, we know that $\mu(H_{\beta_1})$, $\mu(H_{\beta_1}) + \mu(H_{\beta_2})$ and $\mu(H_{\beta_1}) + \mu(H_{\beta_2}) + \mu(H_{\beta_3})$ are nonnegative, so if one of the $\mu(H_{\beta_i})$'s is negative, X_μ must belong to $\mathfrak{g}_{1-10}$, $\mathfrak{g}_{10-1}$ or $\mathfrak{g}_{01-1}$. Since $[\mathfrak{g}_{1-10}, \mathfrak{g}_{01-1}] = \mathfrak{g}_{10-1}$, what remains to be checked is whether Θ is annihilated by any $Z \in \mathfrak{g}_{1-10}$ or $Y \in \mathfrak{g}_{01-1}$. This is equivalent to the four identities

$$[Z_3, T] = (ad(Z)S_{23}) \circ X_{\beta_1}, \tag{5}$$

$$[Y_1, T] = (ad(Y)S_{13}) \circ X_{\beta_2}, \tag{6}$$

$$[Z_1, T] = S_{13} \circ ad(Z)X_{\beta_2}, \tag{7}$$

$$[Y_2, T] = S_{12} \circ ad(Y)X_{\beta_3}, \tag{8}$$

where $[Z_3, T]$, for example, means that we take the bracket of Z only with the terms in T coming from $\mathfrak{g}_{110}$.

Proof of (5). To prove the first identity, we can let $Z = X_{\delta-\beta_2}$ for some $\delta \in \Phi_{110}$. Then

$$[Z_3, T] = \sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\\ \alpha+\beta=\delta+\beta_3}} N_{\beta_3-\alpha,\beta_1-\beta} X_{\beta_1} X_\alpha X_{\alpha'+\delta-\beta_2}.$$

But

$$\begin{aligned} N_{\beta_3-\alpha,\beta_1-\beta} &= -N_{-\alpha,\beta_3+\beta_1-\beta} && (2)\\ &= N_{\beta_2-\alpha,\beta_3-\beta} && (3)\ \text{twice}\\ &= N_{\beta_2-\alpha,\alpha-\delta} && \\ &= N_{\beta_2-\alpha,\delta-\beta_2} && (4)\\ &= N_{\alpha',\delta-\beta_2} && (3). \end{aligned}$$

This allows us to write $[Z_3, T]$ as

$$\begin{aligned} \sum_{\alpha\in\Phi_{011}} N_{\alpha',\delta-\beta_2} X_{\beta_1} X_\alpha X_{\alpha'+\delta-\beta_2} &= \frac{1}{2}\left(ad(Z)\left(\sum_{\alpha\in\Phi_{011}} X_\alpha X_{\alpha'}\right)\right) X_{\beta_1}\\ &= (ad(Z)S_{23}) \circ X_{\beta_1}. \end{aligned}$$

This proves (5). The proof of (6) is similar and will be left to the reader.

Proof of (8). The proofs of (7) and (8) involve the same type of arguments and we will focus on (8). We use the invariance of S_{12} from Lemma 6.5. Let $Y = X_{\theta-\beta_3}$, with $\theta \in \Phi_{011}$. We have $ad(X_\theta)S_{12} = 0$ since $\mathfrak{g}_{121} = 0$. Thus, for $\alpha \in \Phi_{011}$,

$$ad(X_\theta)ad(X_{-\alpha})S_{12} = ad([X_\theta, X_{-\alpha}])S_{12}.$$

If $\alpha \neq \theta$, $[X_{-\theta}, X_\alpha]$ is either zero or a root vector in $\mathfrak{g}_{000}$, thus annihilating S_{12}. Hence

$$\begin{aligned} X_\theta ad([X_\theta, X_{-\theta}])S_{12} &= \textstyle\sum_{\alpha\in\Phi_{011}} X_\alpha ad(X_\theta)ad(X_{-\alpha})S_{12}\\ &= \textstyle\sum_{\alpha\in\Phi_{011},\gamma\in\Phi_{110}} N_{-\alpha,\gamma'} N_{\gamma'-\alpha,\theta} X_\alpha X_{\gamma'-\alpha+\theta} X_\gamma\\ &= \textstyle\sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\gamma\in\Phi_{110}\\ \alpha+\beta+\gamma=\beta_1+\beta_2+\beta_3}} N_{-\alpha,\gamma'} N_{\gamma'-\alpha,\theta} X_\alpha X_{\beta+\theta-\beta_3} X_\gamma. \end{aligned}$$

But $N_{\gamma'-\alpha,\theta} = N_{\beta-\beta_3,\theta} = -N_{\beta,\theta-\beta_3}$ by (2), and $N_{-\alpha,\gamma'} = N_{-\alpha,\alpha+\beta-\beta_3} = N_{\beta_3-\alpha,\alpha+\beta-\beta_3}$ $= -N_{\beta_3-\alpha,-\beta} = -N_{\beta_3-\alpha,\beta_1-\beta}$, where we used successively (3), (4) and (3) again. Thus

$$\begin{aligned} X_\theta ad([X_\theta, X_{-\theta}])S_{12} &= \sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\gamma\in\Phi_{110}\\ \alpha+\beta+\gamma=\beta_1+\beta_2+\beta_3}} N_{\beta_3-\alpha,\beta_1-\beta} N_{\beta,\theta-\beta_3} X_\alpha X_{\beta+\theta-\beta_3} X_\gamma,\\ &= \sum_{\substack{\alpha\in\Phi_{011},\beta\in\Phi_{101},\gamma\in\Phi_{110}\\ \alpha+\beta+\gamma=\beta_1+\beta_2+\beta_3}} N_{\beta_3-\alpha,\beta_1-\beta} X_\alpha [X_\beta, X_{\theta-\beta_3}] X_\gamma\\ &= -[Y_2, T]. \end{aligned}$$

There remains to compute $ad([X_\theta, X_{-\theta}])S_{12}$. We have $[X_\theta, X_{-\theta}] = t_\theta H_\theta$, where t_θ can be computed as follows:

$$t_\theta K(H_\theta, H_\theta) = K([X_\theta, X_{-\theta}], H_\theta) = K(X_\theta, [X_{-\theta}, H_\theta]) = 2K(X_\theta, X_{-\theta}).$$

And since we know that $2K(X_\theta, X_{-\theta}) = -2K(X_{\beta_2}, X_{-\beta_2}) = -K(H_{\beta_2}, H_{\beta_2})$, we get that

$$t_\theta = -\frac{K(H_{\beta_2}, H_{\beta_2})}{K(H_\theta, H_\theta)} = -\frac{(\theta,\theta)}{(\beta_2,\beta_2)} = -\frac{(\theta,\theta)}{(\beta_1,\beta_1)}.$$

Then $ad([X_\theta, X_{-\theta}])S_{12} = t_\theta ad(H_\theta)S_{12}$ is equal to

$$t_\theta \sum_{\gamma\in\Phi_{110}} (\gamma(H_\theta) + \gamma'(H_\theta)) X_\gamma X_{\gamma'} = t_\theta(\beta_1(H_\theta) + \beta_2(H_\theta))S_{12}.$$

But since $\theta \in \Phi_{011}$, $\beta_1(H_\theta) = \theta(H_{\beta_1}) = 0$, while

$$\beta_2(H_\theta) = \frac{(\beta_2,\beta_2)}{(\theta,\theta)}\theta(H_{\beta_2}) = -t_\theta^{-1}.$$

We thus get that $[Y_2, T] = S_{12}[Y, X_{\beta_2}]$, as required. This concludes the proof of identity (8), and hence of Proposition 6.1. $\square$

6.3. Geometric interpretation of the $Y_k(\beta)$'s

Zak defines the Scorza varieties to be the smooth nondegenerate varieties extremal for higher secant defects in the sense that the defect of the i-th secant variety of $X^n \subset \mathbb{P}^N$ is i times the defect δ of the first and the $[\frac{n}{\delta}]$-th secant variety fills the ambient space. He then goes on to classify the Scorza varieties, all of which turn out to be homogeneous [18].

More precisely, the Scorza varieties are given by the projectivization of the rank one elements in the Jordan algebras $\mathcal{J}_r(\mathbb{A})$, where $\mathbb{A}$ is the complexification of the reals, complex numbers or quaternions, or, when $r = 3$, the octonions [3].

Recall that for a simple Lie algebra, the adjoint group has a unique closed orbit in $\mathbb{P}\mathfrak{g}$, the projectivization X_{ad} of the minimal nilpotent orbit. This *adjoint variety* parametrizes the highest root spaces in $\mathfrak{g}$.

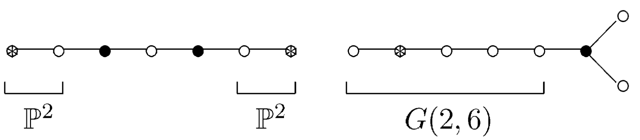
Proposition 6.9. *Let $Z_k \subset X_{ad}$ denote the shadow of a point of the closed orbit $X_k(\beta) \subset \mathbb{P}Y_k(\beta)$. Then the Z_k's are Scorza varieties on the adjoint variety.*

The shadow of a point is defined as follows: let G denote the adjoint algebraic group associated to the Lie algebra $\mathfrak{g}$. We have maps

$$X_k(\beta) = G/Q \xleftarrow{q} G \xrightarrow{p} G/P = X_{ad}$$

and the shadow of $x \in X_k(\beta)$ is the subvariety $p(q^{-1}(x))$ of X_{ad}. Tits [16] showed how to determine these shadows using Dynkin diagrams, and Proposition 6.9 follows from a straightforward case-by-case check.

Example. Let $\mathfrak{g}$ be $\mathfrak{sl}_{l+1}$ or $\mathfrak{so}_{2l}$. On the Dynkin diagram of $\mathfrak{g}$, we let the $*$'s encode the highest weight of the fundamental representation, and the $\bullet$'s encode that of $Y_k(\beta)$. When we suppress the $\bullet$'s, we get weighted diagrams encoding homogeneous varieties, respectively, $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$ and a Grassmannian $G(2, 2k)$ which are two examples of Scorza varieties.



It is natural that the Scorza varieties arrive as subsets of polynomials of degree k on $\mathfrak{g}$ because the determinant on $\mathcal{J}_k(\mathbb{A})$ is a polynomial of degree k. If we take the linear span of Z_k and then take the cone over the degree k hypersurface in $\langle Z_k \rangle$ with vertex a Killing-complement to $\langle Z_k \rangle$, we obtain a hypersurface of degree k in $\mathfrak{g}$. $X_k(\beta)$ parametrizes this space of hypersurfaces and its span gives the space $Y_k(\beta)$.

6.4. Universal dimension formulas

We finally extend our formula for the dimension of the Cartan powers of $\mathfrak{g}$ to obtain a universal formula for the Cartan powers of the $Y_l(\beta)$'s. Again, our approach is based on Weyl's dimension formula: we check whether the relevant integers can be organized into strings whose extremities depend only on Vogel's parameters β and γ. In fact, this really makes sense only in type A, B, D, and in the exceptional cases (excluding $\mathfrak{f}_4$) when $l = 2$. In type C and F_4, there are some strange compensations involving half integers, but the final formula holds in all cases.

We will just give the main statements leading to Theorem 6.10 below. Let $\sigma_l = \beta_1 + \cdots + \beta_l$ denote the highest weight of $Y_l(\beta)$. Let $\Phi_{l,i}$ denote the set of positive roots γ such that $\sigma_l(H_\gamma) = i$. By Lemma 6.2, we have

$$\begin{aligned}
\#\Phi_{l,1} &= 2l(\gamma + 2\beta - 4 - \beta l),\\
\#\Phi_{l,2} &= \beta\frac{l(l-1)}{2} + l,\\
\#\Phi_{l,i} &= 0 \qquad \text{for } i > 2.
\end{aligned}$$

Facts. (1) The values of $\rho(H_\gamma)$, for γ in $\Phi_{l,2}$, can be organized into l intervals $[\gamma - (2l-i-3)\beta/2 - 3, \gamma - (i-3)\beta/2 - 3]$, where $1 \leqslant i \leqslant l$.

(2) The values of $\rho(H_\gamma)$, for γ in $\Phi_{l,1}$, can be organized into $3l$ intervals $[\gamma/2 - (i-1)\beta/2, \gamma/2 - (l-i-2)\beta/2 - 3]$, $[i\beta/2, \gamma - (l+i-4)\beta/2 - 3]$ and $[(i-1)\beta/2 + 1, \gamma - (l+i-3)\beta/2 - 4]$, with $1 \leqslant i \leqslant l$.

Applying the Weyl dimension formula, we obtain the following:

Theorem 6.10.

$$\dim(Y_l(\beta))^{(k)} = \prod_{i=1}^{l} \frac{\binom{2k+\gamma-(i-3)\beta/2-3}{2k}\binom{k+\gamma-(l+i-3)\beta/2-3}{k}\binom{k+\gamma-(l+i-4)\beta/2-4}{k}\binom{k+\gamma/2-(l-i-2)\beta/2-3}{k}}{\binom{2k+\gamma-(2l-i-3)\beta/2-4}{2k}\binom{k+i\beta/2-1}{k}\binom{k+(i-1)\beta/2}{k}\binom{k+\gamma/2-(i-1)\beta/2-1}{k}}.$$

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