1. More on $SL_n$ and its representations

1.1. Recap on the irreducible $\mathfrak{sl}_n$ modules. The $n-1$ root spaces that span $\mathfrak{g}_1$ give rise to $n-1$ distinguished roots in $\mathfrak{h}^*$ which will be called simple roots, namely $L_i - L_{i+1}$, $1 \leq i \leq n-1$, and they give rise to $n-1$ distinguished copies of $\mathfrak{sl}_2$, namely $\mathbb{C}\{E_i^{\pm 1}, E_{i+1}^\pm E_i^{\mp 1}, E_{i+1}^\pm \}$, such that when the $i$-th lowering operator acts, it sends $L_i$ to $L_{i+1}$ so, recalling that $\omega_i := L_1 + \cdots + L_i$, it acts by

$$E_i^{\pm 1}(\lambda_1\omega_1 + \cdots + \lambda_n\omega_n) = \lambda_1\omega_1 + \cdots + \lambda_{i-1}\omega_{i-1} + (\lambda_i - 1)\omega_i + (\lambda_i - 2)\omega_{i-1} + (\lambda_{i+1} + 1)\omega_{i+1} + \lambda_{i+2}\omega_{i+2} + \cdots + \lambda_{n-1}\omega_{n-1}.$$ 

In pictures, acting by the fourth lowering operator: goes to

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a1   a1   a3   a4   a5   a6
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a1   a1   a3+1 a4-2 a5+1 a6
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By the same argument as for $\mathfrak{sl}_2$ we conclude i) the $\lambda_i$ are integers in any representation, ii) in any representation there is a “highest weight” $\lambda$, iii) that all weights are contained in the convex hull of $\pm \lambda_1\omega_1 + \cdots + \pm \lambda_{n-1}\omega_{n-1}$, and iv) if the representation is irreducible, there is a unique highest weight line. Let $\Lambda_W$ denote the lattice generated by the $L_i$, the weight lattice and $\Lambda_R$ the root lattice, which is generated by the $L_i - L_j$. The weights of any finite dimensional $\mathfrak{sl}_n$-module are all in $\Lambda_W$, and if the module is irreducible, all the weights are congruent modulo $\Lambda_R$.

For every $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}_{\geq 0}$ there exists a unique irreducible module with highest weight $\lambda$. To see this, thanks to Cartan products, it will be enough to construct the representations $V_{\omega_i}$. Note that $e_1 \wedge e_2 \wedge \cdots \wedge e_i \in \Lambda^i V$ is a highest weight vector and has weight $\omega_i$, so we have $V_{\omega_i}$ is realized as $\Lambda^i V$. (The uniqueness proof is identical to that for $\mathfrak{sl}_2$.) Note that the weights of $V_{\omega_i}$ are all of the form $L_{j_1} + \cdots + L_{j_i}$ where $1 \leq j_1 < \cdots < j_i \leq n$ and all have multiplicity one, as $e_{j_1} \wedge \cdots \wedge e_{j_i}$ form a basis of $\Lambda^i V$ and are weight vectors of those weights.

One can similarly describe $V_{d\omega_1}$ as $S^d V$ and the weights are all of the form $L_{j_1} + \cdots + L_{j_i}$ where $1 \leq j_1 \leq \cdots \leq j_i \leq n$ and all have multiplicity one, as $e_{j_1} \cdots e_{j_i}$ form a basis of $S^d V$ and are weight vectors of those weights.

The two examples above are pathological - most irreducible representations of $\mathfrak{sl}_n$ for $n > 2$ will have weights with multiplicity greater than one. See FH and the weights of $V_{2\omega_1 + \omega_2}$ for $\mathfrak{sl}_3$ and notice if instead of $e_3$ one uses $e_1 \wedge e_2$ etc.. one gets many generalizations.
1.2. Geometry. Recall that if \( v_\lambda \in V_\lambda \) is a highest weight vector, we considered the homogeneous space \( SL(V) \cdot [v_\lambda] \subset \mathbb{P}V_\lambda \) which we saw was exactly the zero set of the polynomials \( V_{2\lambda} \subset S^2 V_\lambda^* \). More generally, the same proof shows that \( I_d(SL(V) \cdot [v_\lambda]) = V_{d\lambda} \subset S^d V_\lambda^* \).

**Exercise 1.2.1:** Note that the orbit \( SL(V) \cdot [v_\lambda] \) is independent of any of our choices. Show that it admits the interpretation as the space of possible highest weight lines, under all the choices we make.

If \( \lambda = d\omega_1 \), then \( SL(V) \cdot [v_\lambda] = v_d(\mathbb{P}V) := \mathbb{P}\{ P \in S^d V \mid P = \ell^d \text{ for some } \ell \in V \} \) is called the Veronese variety. Note that as a manifold (and abstract algebraic variety) it is just \( \mathbb{P}V \).

If \( \lambda = \omega_k \), we saw \( SL(V) \cdot [v_\lambda] = \mathbb{P}\{ X \in \Lambda^k V \mid X = v_1 \wedge \cdots \wedge v_k \text{ for some } v_j \in V \} \) because under \( SL(V) \) we can map \( e_1, \ldots, e_k \) to any \( k \) linearly independent vectors in \( V \). This orbit is called the Grassmann variety or Grassmannian and is denoted \( G(k,V) \). The Grassmannian admits the geometric interpretation as the set of \( k \)-planes through the origin in \( V \) by the following exercise:

**Exercise 1.2.2:** Given a \( k \)-plane \( E \), let \( v_1, \ldots, v_k \) be a basis of \( E \). Show that for any \( w_1, \ldots, w_k \in V \), that \( [v_1 \wedge \cdots \wedge v_k] = [w_1 \wedge \cdots \wedge w_k] \), if and only if \( w_1, \ldots, w_k \) is a second basis of \( E \).

The identical argument as for the Veronese shows that \( I_2(G(k,V)) = V_{2\omega_k} \subset S^2(\Lambda^k V^*) \) and that the Grassmannian is exactly the zero set of these equations. (Later we will show the stronger assertion that these equations generate the ideal.)

So we need to determine the decomposition of \( S^2(\Lambda^k V) \). (I drop the duals from the notation for simplicity.)

**Exercise 1.2.3:** Show that we have well defined map \( S^2(\Lambda^2 V) \to \Lambda^4 V \) given by \( \phi \psi \mapsto \phi \wedge \psi \) and under this map \((v \wedge w)^2 \mapsto 0\).

**Exercise 1.2.4:** More generally show that we have well defined maps \( S^2(\Lambda^k V) \to \Lambda^{k-2} V \otimes \Lambda^{k+2} V \) given by (the hat denotes an omitted element)

\[
(v_1 \wedge \cdots \wedge v_k)(w_1 \wedge \cdots \wedge w_k) \mapsto \\
\sum_{i<j}(-1)^{i+j}v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k \otimes v_i \wedge \cdots \wedge v_j \wedge w_1 \wedge \cdots \wedge w_k \wedge w_i \wedge \cdots \wedge w_j.
\]

**Exercise 1.2.5:** Show that even more generally we have well defined maps \( S^2(\Lambda^k V) \to \Lambda^{k-2p} V \otimes \Lambda^{k+2p} V \) with \((v_1 \wedge \cdots \wedge v_k)^2 \) in the kernel. In particular, assuming \( k \leq n - k \), we have

\[
S^2(\Lambda^k V) \cong \bigoplus_{i=0}^{\lfloor k/2 \rfloor} V_{\omega_{k-2i}} \otimes V_{\omega_{k+2i}}.
\]

Show that in fact equality holds.

**Remark**—more pictures of marked Dynkin diagrams were given in class.**

1.3. Other orbits. For \( v_{d\omega_i} \in V_{d\omega_i} \subset S^d(\Lambda^1 V) \), it is easy to check that \((e_1 \wedge \cdots \wedge e_i)^d \) is a highest weight vector and that \( SL(V) \cdot [v_{d\omega_i}] = v_d(G(i,V)) \subset \mathbb{P}V_{d\omega_i} \), the image of the Grassmannian under the Veronese re-embedding. The orbit must be contained in an irreducible module because it is the orbit of a highest weight vector.

**Exercise 1.3.1:** Show that \( \langle SL(V) \cdot [v_{d\omega_i}] \rangle = V_{d\omega_i} \).

Now consider, for \( i < j, SL(V) \cdot [v_{\omega_i + \omega_j}] \subset \mathbb{P}V_{\omega_i + \omega_j} \subset (\mathbb{P}V_{\omega_i} \otimes V_{\omega_j}) \). The highest weight vector is \( e_1 \wedge \cdots \wedge e_i \otimes e_1 \wedge \cdots \wedge e_j \).

**Exercise 1.3.2:** Show that \( SL(V) \cdot [v_{\omega_i + \omega_j}] = Flag_{i,j}(V) := \{ (E,F) \in G(i,V) \times G(j,V) \mid E \subset F \} \). More generally show that for \( i < j < k, SL(V) \cdot [v_{\omega_i + \omega_j + \omega_k}] = Flag_{i,j,k}(V) \) etc.
An important special case is $\rho := \omega_1 + \cdots + \omega_{n-1}$ where one gets the variety of complete flags. It is useful to look at the stabilizers: write $G_{[v]} \subset SL(V)$ for the stabilizer of $[v] \in \mathbb{P}W$. Then

$$G_{[e_1, \ldots, e_k]} = \{ A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mid \det(A) = 1 \}$$

where the blocking is $k \times n - k$ and $G_{v_i}$ is just the set of upper triangular matrices with determinant one, which comes up so often it gets its own name $B$. Applying the Chinese remainder theorem in the case $\mathbb{Q} \subset \mathbb{C}$ states that in a ring $R$ with unit, if $I_1, \ldots, I_k$ are ideals such that for all $i \neq j$, $I_i + I_j = R$, then for all $b_1, \ldots, b_k \in R$, there exist $b \in R$ such that $b \equiv b_i \mod I_i$ (and $b$ is unique up to elements of $\cap I_i$).

Recall that the Chinese remainder theorem states that in a ring $R$ with unit, if $I_1, \ldots, I_k$ are ideals such that for all $i \neq j$, $I_i + I_j = R$, then for all $b_1, \ldots, b_k \in R$, there exist $b \in R$ such that $b \equiv b_i \mod I_i$ (and $b$ is unique up to elements of $\cap I_i$).

Applying the Chinese remainder theorem in the case $R = \mathbb{C}[t]$, $I_j$ is the ideal generated by $(t - \lambda_j)^{m_j}$, and $b_i = \lambda_i$ we see there is a polynomial $p(t)$ such that for all $i$ we may write $p(t) = \lambda_i + R_i(t)(t - \lambda_i)^{m_i}$. Then setting $X_s = p(X)$ we see $X_s$ acts as $\lambda_i \text{Id}$ on $V_i$. Setting $q(t) = t - p(t)$ and $X_n = q(X)$, all the assertions of the proposition are clear except for the uniqueness.

**Exercise 1.3.3:** Consider the double fibration picture

$$Flag_{i,j}(V) \hookrightarrow G(i, V) \rightrightarrows G(j, V)$$

call the projections $\pi_i$ and $\pi_j$. Let $E \in G(i, V)$, what is $\pi^{-1}_i(E)$? What is $\pi_j(\pi_i(E))$?

Such stabilizers are generally denoted by “$P$” for parabolic subgroup, where $P \subset G$ is parabolic if $G/P$ is compact. Such groups are not even semi-simple, but they do have a “natural” semi-simple subgroup, which for $G(k, V)$ is just $SL_k \times SL_{n-k}$. (It is even better to study the group $G_0 := S(GL_k \times GL_{n-k})$ which is reductive with a one-dimensional center.) Now $T_{[V_\lambda]} SL(V) [V_\lambda] = T_{[Id]} G/P = g/p$ and thus is naturally acted on by $P$ and $p$.

**Exercise 1.3.4:** Show that in the case of the Grassmannian $G(k, V)$, $T_E G(k, V) \simeq E^* \otimes V/E$ as an $SL(E) \times SL(V/E)$-module.

2. Jordan decomposition, Casimirs, and characterizations of solvable and semi-simple Lie algebras

2.1. Jordan canonical form.

**Proposition 2.1.1** (Jordan decomposition). Let $X \in \text{End}(V)$. Then

(1) There exist unique $X_s, X_n \in \text{End}(V)$ such that

(a) $X = X_s + X_n$.

(b) $X_s$ is diagonalizable (a diagonalizable endomorphism is called semi-simple), i.e., all roots of its minimal polynomial are distinct.

(c) $X_n$ is nilpotent.

(d) $X_s, X_n$ commute.

(2) There exist polynomials $p(t), q(t)$ in one variable such that $X_s = p(X)$ and $X_n = q(X)$.

(In particular, $X_s, X_n$ commute with any endomorphism commuting with $X$.)

(3) If $W \subset U \subset V$ and $X(U) \subset W$, then $X_s, X_n : U \to W$.

**Proof.** Write the characteristic polynomial of $X$ as $\det(xI - X) = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$ (i.e., the eigenvalues of $X$ are the $\lambda_i$ and the multiplicity of $\lambda_i$ is $m_i$). Let $V_i = \ker(X - \lambda_i \text{Id})^{m_i}$ denote the generalized eigenspace of $\lambda_i$, so we have a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$, with each $V_i$ stable under $X$ and $cp_{X|V_i}(t) = (t - \lambda_i)^{m_i}$.
To see the uniqueness, let $X = Y_s + Y_n$ be another decomposition, so $Y_s - X_s = X_n - Y_n$ and note that $Y_s$ and $X_s$ commute. Linear combinations of commuting semi-simple endomorphisms are semi-simple. But $X_n, Y_n$ also commute and linear combinations of commuting nilpotent endomorphisms are nilpotent. But an endomorphism that is both semi-simple and nilpotent is zero. □

2.2. **Invariant bilinear forms on Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra, $V$ a $\mathfrak{g}$-module, write $\mu : \mathfrak{g} \to \mathfrak{gl}(V)$. Define a bilinear form $B_V \in S^2 \mathfrak{g}^*$ by $B_V(X,Y) := \text{trace}(\mu(X) \circ \mu(Y))$. It is clear $B_V$ is symmetric and bilinear. It is also $\text{Ad}$-invariant, i.e. annihilated by $\mathfrak{g}$, as

$$(ad(Z)B_V)(X,Y) = B_V(ad(Z)X,Y) + B_V(X,ad(Z)Y)$$

$$= \text{trace}(\mu(Z)\mu(X)\mu(Y) - \mu(X)\mu(Z)\mu(Y)) + \text{trace}(\mu(X)\mu(Z)\mu(Y) - \mu(X)\mu(Y)\mu(Z))$$

$$= 0.$$ 

The last line follows as $\text{trace}(\mu(Z)(\mu(X)\mu(Y)))) = \text{trace}((\mu(X)\mu(Y))\mu(Z))$. When $V = \mathfrak{g}$ and $\mu = \text{ad}$ we write $B_\mathfrak{g} = B$ and call it the **Killing-Cartan form**.

2.3. **Characterizations of solvable and semi-simple Lie algebras.** Read the exposition in FH.

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