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REMARKS ON
“LIE ALGEBRA COHOMOLOGY AND THE GENERALIZED BOREL-WEIL THEOREM”, BY B. KOSTANT

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In Kostant’s paper, which appears in this issue of the Annals, that author studies the cohomology of certain subalgebras of a semi-simple Lie algebra. One of his results is Theorem 5.14 from which he is able to deduce the Borel-Weil theorem as generalized by R. Bott. I would like to point out that the whole reasoning by which Kostant makes this deduction effective can be inverted; therefore his result is equivalent to Borel-Weil-Bott theorem.

Next, following an idea of Bott, he proceeds to prove H. Weyl’s well-known character formula. The point is to interpret each character as a ratio of two Euler-Poincaré characteristics which amount respectively for the numerator and the denominator of Weyl’s formula. Unfortunately, as pointed out by Kostant himself, one needs a very particular case of Weyl’s formula in the course of proof of Theorem 5.14. We would like to sketch a proof of Theorem 5.14 which does not use even this particular case of Weyl’s formula. The point is to give a direct proof of Lemma 5.12 omitting all of §§ 5.8 to 5.11. Such a proof follows. (We do not repeat all notations).

Δ being the set of all roots, let Φ be any subset of Δ; by \(\langle \Phi \rangle\) we mean the sum of all roots belonging to Φ and \(g = 1/2\langle \Delta_+ \rangle\) (\(\Delta_+\) is the set of all positive roots). There is a one-to-one correspondence between the set of all subsets Φ of \(\Delta_+\) and the set of all subsets \(\Psi\) of Δ such that Δ be a disjoint union of \(\Psi\) and \(-\Psi\); this correspondence is expressed by the formulas \(\Phi = \Psi \cap \Delta_+\) and \(\Psi = \Phi \cup -(\Delta_+ \cap C\Phi)\); furthermore, by an easy computation we get

\[
g - \langle \Phi \rangle = -1/2\langle \Psi \rangle
\]

when \(\Phi\) and \(\Psi\) are so related.

Let \(\mathfrak{m}\) be the Lie subalgebra of the semi-simple Lie algebra \(\mathfrak{g}\) generated by the Cartan subalgebra \(\mathfrak{h}\) and the root vectors belonging to positive roots. If \(V^\lambda\) is the space of an irreducible representation \(\pi\) of \(\mathfrak{g}\) with maximal weight \(\lambda\), one considers the natural representation \(\zeta\) of \(\mathfrak{h}\) on the space \(\Lambda m^* \otimes V^\lambda\). Using the well-known basis of an exterior algebra, one sees the weights of \(\zeta\) are of the form:
where $\mu$ is a weight of $\pi$ and $\Phi$ is any subset of $\Delta_+$; furthermore the multiplicity of $\xi$ is the sum of the multiplicities $m_{\mu}$ (for $\mu$ considered as a weight of $\pi$) extended over all decompositions of $\xi$ in the form (2).

I claim

\begin{equation}
| g + \lambda | \geq | g + \xi | \tag{3}
\end{equation}

for any weight $\xi$ of $\zeta$. Using (1) and (2) we get

\begin{equation}
g + \xi = \mu - 1/2 \langle \Psi \rangle \tag{4}
\end{equation}

Since the set of weights of $\pi$ is invariant under the Weyl group $W$, so it is for the set of linear forms $g + \xi$ on $\mathfrak{h}$:

\begin{equation}
s(g + \xi) = s \cdot \mu - 1/2 \langle s \cdot \Psi \rangle = g + s \cdot \mu - \langle \Phi(s) \rangle \tag{5}
\end{equation}

where $\Phi(s) = s \cdot \Psi \cap \Delta_+$. We can therefore find an $s$ in $W$ so that $s(g + \xi)$ is dominant. As is well-known, $s \cdot \mu$ is equal to $\lambda - \sum_i \alpha_i$ with positive roots $\alpha_i$ and non-negative integers $m_i$; it implies:

\begin{equation}
s(g + \xi) = (g + \lambda) - \sum_i m_i \cdot \alpha_i, \tag{6}
\end{equation}

and finally

\begin{align*}
| g + \lambda |^2 &= | s(g + \xi) |^2 + | \sum_i m_i \cdot \alpha_i |^2 + \sum_i m_i \langle s(g + \xi), \alpha_i \rangle \\
&= | g + \xi |^2 + | \sum_i m_i \cdot \alpha_i |^2 + \sum_i m_i \langle s(g + \xi), \alpha_i \rangle .
\end{align*}

Since $s(g + \xi)$ is dominant, the scalar product $\langle s(g + \xi), \alpha \rangle$ is non-negative for each positive root $\alpha$. It follows formula (3) immediately.

From this deduction of formula (3), one sees sign "equal" can occur only for all $m_i$ equal to 0, that is $s(g + \xi) = g + \lambda$ or $\xi = \xi_s$ with

\begin{equation}
\xi_s = s^{-1}(g + \lambda) - g. \tag{7}
\end{equation}

Furthermore $\lambda$ is dominant and $\langle g, \alpha \rangle > 0$ for any positive root $\alpha$, so that $\langle g + \lambda, \alpha \rangle > 0$ under the same assumptions; as is well-known, this implies $s(g + \lambda) \neq g + \lambda$ for $s \neq 1$ and there is a unique $s$ in $W$ for which (7) holds. The map $s \rightarrow \xi_s$ is bijective from $W$ to the set of all weights $\xi$ of $\zeta$ such that $| g + \xi | = | g + \lambda |$.

It remains to show that the weight $\xi_s$ given by (7) has multiplicity one. Since $\lambda$ occurs with multiplicity one in $\pi$, it is sufficient to show $\xi_s$ has a unique decomposition in the form (2) and $\lambda = \mu$ in this decomposition. But (2) implies (5) and using (7) one gets:

\begin{equation}
\lambda = s \cdot \mu - \langle \Phi(s) \rangle. \tag{8}
\end{equation}

Recalling $s \cdot \mu = \lambda - \sum_i m_i \cdot \alpha_i$ with non-negative integers $m_i$, this is
possible only if all $m_i$ are 0 and $\Phi(s)$ is empty, that is $\lambda = \mu$ and $s \cdot \Psi \cap \Delta_+$ empty which amounts to $\Psi = s^{-1} \cdot \Delta_-$, or finally $\Phi = \Delta_+ \cap s^{-1} \cdot \Delta_-$. This achieves the proof.

We have proved all of Lemma 5.12, the last assertion in it being trivial any way. This concludes our task.