Communications in
Mathematical
Physics

The Spectra of Quantum States and the Kronecker Coefficients of the Symmetric Group

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Abstract: Determining the relationship between composite systems and their subsystems is a fundamental problem in quantum physics. In this paper we consider the spectra of a bipartite quantum state and its two marginal states. To each spectrum we can associate a representation of the symmetric group defined by a Young diagram whose normalised row lengths approximate the spectrum. We show that, for allowed spectra, the representation of the composite system is contained in the tensor product of the representations of the two subsystems. This gives a new physical meaning to representations of the symmetric group. It also introduces a new way of using the machinery of group theory in quantum informational problems, which we illustrate by two simple examples.

I. Introduction

In 1930, Weyl observed with dry humour that the "group pest" seemed to be here to stay ([10], preface to second German edition). The theory of representations of groups, which he did so much to develop, is indeed a firmly established component of modern physics, appearing wherever the relation of a composite system to its parts is investigated. The aim of this paper is to derive a novel connection between certain representations and the properties of composite quantum systems.

Suppose a quantum system consists of two parts, A and B, and let ρ^{AB} be a density operator on the composite system AB. The states ρ^A and ρ^B obtained by tracing out the subsystems B and A, respectively, are constrained by the fact that they are derived from a common state. For instance, subadditivity and the triangle inequality are informational inequalities that relate the von Neumann entropies (the Shannon entropies of the spectra) of ρ^{AB}, ρ^A and ρ^B. Even more fundamentally, however, one can ask what constraints there are on the spectra of ρ^A and ρ^B once one knows the spectrum of ρ^{AB}. We prove here a theorem that relates this problem to certain representations of the unitary and symmetric groups.

A familiar example of a composite system is two particles, one with spin j_1 and the other with spin j_2. The addition of their angular momenta can be described in terms of representations of SU(2); the product of two representations, one for each subsystem, can be expressed as a sum of representations on the total system. This is the familiar Clebsch-Gordan series, whose coefficients have been much studied and can be readily calculated. There is an analogous expansion of the product of two representations of the symmetric group S_k on k elements. The coefficients appearing in this alternative Clebsch-Gordan series are known as Kronecker coefficients, and their evaluation is more difficult: no simple algorithm is known at present.

To state our result, we need a little notation. As we shall see shortly, every irreducible representation of the symmetric group S_k can be labelled by an ordered partition $\lambda = (\lambda_1, \dots, \lambda_q)$ of k; i.e. a set of non-negative integers λ_i with $\lambda_{i+1} \leq \lambda_i$ and $\sum \lambda_i = k$. Let $\bar{\lambda}$ denote $(\frac{\lambda_1}{k}, \dots, \frac{\lambda_q}{k})$ and let $g_{\lambda\mu\nu}$ denote the Kronecker coefficient that counts the number of times (possibly zero) that the representation labelled by λ appears in the product of those labelled by μ and ν. Finally, let $\mathrm{spec}(\rho)$ denote the spectrum of ρ. Then our main result is that, given a density operator ρ^{AB} with $\mathrm{spec}(\rho^{AB}) = \bar{\lambda}$, $\mathrm{spec}(\rho^A) = \bar{\mu}$ and $\mathrm{spec}(\rho^B) = \bar{\nu}$, there is a sequence λ_j, μ_j, ν_j with non-zero $g_{\lambda_j\mu_j\nu_j}$ such that $\bar{\lambda}_j, \bar{\mu}_j$ and $\bar{\nu}_j$ converge to $\mathrm{spec}(\rho^{AB})$, $\mathrm{spec}(\rho^A)$ and $\mathrm{spec}(\rho^B)$, respectively.

II. Young Diagrams and the Spectrum of a Density Operator

In this section we give a brief description of representations of the symmetric group S_k and the special unitary group in d dimensions, $SU(d)$, and review a theorem by Keyl and Werner [6], which will play a key role in proving our main result.

If $\mathbb{C}^d$ denotes a d-dimensional complex vector space, S_k operates on $(\mathbb{C}^d)^{\otimes k}$ by

$$\pi\{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}\} = e_{i_{\pi^{-1}(1)}} \otimes e_{i_{\pi^{-1}(2)}} \otimes \dots \otimes e_{i_{\pi^{-1}(k)}}, \tag{1}$$

for $\pi \in S_k$, where the $e_1, \dots e_d$ are elements of some basis of $\mathbb{C}^d$. The group $SU(d)$ acts by

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k} \to U e_{i_1} \otimes U e_{i_2} \otimes \dots \otimes U e_{i_k}, \tag{2}$$

for $U \in SU(d)$.

These actions of S_k and $SU(d)$ on $(\mathbb{C}^d)^{\otimes k}$ define representations of each group, but both representations are reducible. Their irreducible components can be constructed as follows. Let us write $\lambda \vdash k$ to mean that λ is an ordered partition with $\sum \lambda_i = |\lambda| = k$. This can be depicted by a *Young frame*, which consists of d rows, the ith row having λ_i boxes in it. A *Young tableau* T is obtained from a frame by filling the boxes with the numbers 1 to k in some order, with the constraint that the numbers in each row increase on going to the right and the numbers in each column increase downwards.

To each tableau T, we associate the *Young symmetry operator* $e(T)$ given by

$$e(T) = \left(\sum_{\pi \in \mathcal{C}(T)} \mathrm{sgn}\,(\pi)\pi \right)\left(\sum_{\pi \in \mathcal{R}(T)} \pi \right), \tag{3}$$

where $\mathcal{R}(T)$ and $\mathcal{C}(T)$ are sets of permutations of S_k, $\mathcal{R}(T)$ being those that are obtained by permuting the integers within each row of T, and $\mathcal{C}(T)$ those obtained by permuting integers within each column of T [10].

Each $e(T)$ satisfies $e(T)^2 = re(T)$ for some integer r, so $e(T)/r$ is a projection which we denote by $p(T)$. The action of $SU(d)$ on the image subspace of $p(T)$ in $(\mathbb{C}^d)^{\otimes k}$ gives an irreducible representation of $SU(d)$. If T' is another tableau of the same frame, the representations of $SU(d)$ are equivalent (under the permutation that takes T to T'). Thus the irreducible representations of $SU(d)$ are labelled by Young frames, or equivalently, by partitions $\lambda \vdash k$.

Now pick a vector v in the subspace defined by $p(T)$, and apply all elements $\pi \in S_k$ to it. The subspace of $(\mathbb{C}^d)^{\otimes k}$ spanned by $\{\pi v : \pi \in S_k\}$ defines an irreducible representation of S_k. Distinct frames yield distinct representations, so we can also label the irreducible representations of S_k by partitions $\lambda \vdash k$.

From the above construction, it can be shown that the subspaces $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ of the irreducible representations of S_k and $SU(d)$, respectively, are related in the following elegant manner:

$$\left(\mathbb{C}^d\right)^{\otimes k} = \bigoplus_{\lambda \vdash k} \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda. \tag{4}$$

This is sometimes called the Weyl-Schur duality of S_k and $SU(d)$.

A systematic way to generate $\mathcal{V}_\lambda$ for a tableau T is to apply $p(T)$ to all vectors $v = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}$, where we identify the jth component of the tensor product with the jth box in the numbering of the tableau T. If we count the number of times each basis element e_i occurs in v, this defines a partition $\nu \vdash k$. We say ν is *majorized* by λ, and write $\nu \prec \lambda$, if $\sum_{i=1}^{q} \nu_i \leq \sum_{i=1}^{q} \lambda_i$ for $q = 1, \ldots, d-1$ and $\sum_{i=1}^{d} \nu_i = \sum_{i=1}^{d} \lambda_i$. The vector v will project to zero under $p(T)$ unless $\nu \prec \lambda$, since otherwise there must be two boxes in the same column of T, with numberings i and j, for which $e_i = e_j$. In particular, for any Young diagram with more than d rows, $\mathcal{V}_\lambda = 0$.

The dimensions of $\mathcal{V}_\lambda$ and $\mathcal{U}_\lambda$ are given by [5]

$$\dim \mathcal{V}_\lambda = \frac{\prod_{i<j}(\lambda_i - \lambda_j - i + j)}{\prod_{m=1}^{d-1} m!} \tag{5}$$

and

$$\dim \mathcal{U}_\lambda = \frac{k!}{\prod_i \text{hook}(i)}, \tag{6}$$

where the index i in the latter formula runs over all boxes in the Young diagram of λ, and hook(i) is the hook-length of box i, i.e. the number of boxes vertically below i and to the right of i within the diagram, including box i. Useful bounds for these dimensions are

$$\dim \mathcal{V}_\lambda \leq (k+1)^{d(d-1)/2} \tag{7}$$

and

$$\frac{k!}{\prod_i (\lambda_i + d - i)!} \leq \dim \mathcal{U}_\lambda \leq \frac{k!}{\prod_i \lambda_i!}. \tag{8}$$

A remarkable connection between Young frames and density operators was discovered by Keyl and Werner [6] (see also R. Alicki, S. Rudnicki and S. Sadowski "Symmetry properties of product states for the system of N n-level atoms" J. Math. Phys. vol.29 no.5 pp. 1158–1162 (1988)). Suppose ρ is a density operator with spectrum spec(ρ). Keyl

and Werner showed that, for large k, the quantum state $\rho^{\otimes k}$ will project with high probability into the Young subspaces $\lambda \vdash k$ such that $\bar{\lambda}$ approximates $\text{spec}(\rho)$. Their proof was somewhat elaborate. A succinct argument was found by Hayashi and Matsumoto [4], and we give it here, correcting an algebraic slip in their derivation.

Theorem 1. *Let ρ be a density operator with spectrum $r = \text{spec}(\rho)$, and let P_λ be the projection onto $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$. Then*

$$\text{tr } P_\lambda \rho^{\otimes k} \leq (k+1)^{d(d-1)/2} \exp\left(-kD(\bar{\lambda}||r)\right) \tag{9}$$

with $D(p||q) = \sum_i p_i (\log p_i - \log q_i)$ the Kullback-Leibler distance of two normalised probability distributions p and q. Note that $D(p||q) = 0$ if and only if $p = q$.

Proof. In the procedure for generating $\mathcal{V}_\lambda$ described above, we can choose the eigenvectors of ρ as a basis for $\mathbb{C}^d$. When eigenvectors of $\rho^{\otimes k}$ are projected onto $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$, the result is non-zero only if $\mu \prec \lambda$, where μ_i is the multiplicity of the i-th basis element in the eigenvector. The 'surviving' eigenvalues $\prod_i r_i^{\mu_i}$, are therefore smaller than $\prod_i r_i^{\lambda_i}$.

Using the bounds (7) and (8), it follows that

$$\text{tr } P_\lambda \rho^{\otimes k} \leq \dim \mathcal{U}_\lambda \dim \mathcal{V}_\lambda \prod_i r_i^{\lambda_i} \tag{10}$$

$$\leq (k+1)^{d(d-1)/2} \frac{k!}{\prod_i \lambda_i!} \prod_i r_i^{\lambda_i} \tag{11}$$

$$\leq (k+1)^{d(d-1)/2} \exp\left(-kD(\bar{\lambda}||r)\right). \tag{12}$$

This completes the proof. □

Corollary 1. *If ρ is a density operator with spectrum $r = \text{spec}(\rho)$,*

$$\text{tr } P_X \rho^{\otimes k} \leq (k+1)^{d(d+1)/2} \exp(-k \min_{\lambda \vdash n: \bar{\lambda} \in \mathcal{S}} D(\bar{\lambda}||r)), \tag{13}$$

where $P_X := \sum_{\lambda \vdash k: \bar{\lambda} \in \mathcal{S}} P_\lambda$ for a set of spectra $\mathcal{S}$.

This follows from the theorem if we simply pick the Young frame with the slowest convergence and multiply it by the total number of possible Young frames with k boxes in d rows. This number is certainly smaller than $(k+1)^d$.

Let $\mathcal{B}_\epsilon(r) := \{r' : \sum |r'_i - r_i| < \epsilon\}$ be the ϵ-ball around the spectrum r. If we take $\mathcal{S}$ to be the complement of $\mathcal{B}_\epsilon(r)$, it becomes clear that for large k, $\rho^{\otimes k}$ will project into a Young subspace λ with $\bar{\lambda}$ close to r with high probability. More precisely

Corollary 2. *Given an operator ρ with spectrum $r = \text{spec}(\rho)$, and given $\epsilon_1 > 0$, let $P_X = \sum_{\lambda \vdash k: \bar{\lambda} \in \mathcal{B}_{\epsilon_1}(r)} P_\lambda$. Then for any $\epsilon_2 > 0$ there is a $k_0 > 0$ such that for all $k \geq k_0$,*

$$\text{tr } P_X \rho^{\otimes k} > 1 - \epsilon_2. \tag{14}$$

III. The Content Expansion

Suppose now we have a bipartite system AB made up of systems A and B with spaces $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. $SU(mn)$ thus acts on $\mathbb{C}^{mn}$, the space of AB, and hence on the k-fold tensor product $(\mathbb{C}^{mn})^{\otimes k}$ according to Eq. (2). Similarly, $SU(m)$ and $SU(n)$ act on A and B, respectively, which gives an action of $SU(m) \times SU(n)$ on AB. If $\mathcal{R}_\lambda$ is an irreducible representation of $SU(mn)$ on the Young subspace λ of AB, its restriction to $SU(m) \times SU(n)$ is not necessarily irreducible, and can generally be expressed as a sum of terms $\mathcal{R}_\mu \otimes \mathcal{R}_\nu$, where $\mathcal{R}_\mu$ and $\mathcal{R}_\nu$ are irreducible representions of $SU(m)$ and $SU(n)$, respectively. We call this sum the *content expansion* of λ, borrowing some terminology from [3, 5].

In the remainder of this section we follow [8] closely. The content expansion can be conveniently described in terms of the characters of the underlying representations. The conjugacy class of a unitary matrix in $SU(d)$, for some dimension d, is given by its d eigenvalues $x_1, \dots, x_d$. The character s_λ of the representation $\mathcal{V}_\lambda$ of $SU(d)$ is therefore a function of $x_1, \dots, x_d$. Define the homogeneous power sums by $h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$. Then

$$s_\lambda(x) = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq d}. \tag{15}$$

The polynomial $s_\lambda(x)$ is called the *Schur function* or *S-function* of λ. The s_λ form a basis for the symmetric polynomials in d variables of degree $|\lambda|$.

Now take the character $s_\lambda(z) = s_\lambda(z_1, \dots, z_{mn})$ of the representation λ of $SU(mn)$. When restricted to $SU(m) \times SU(n)$, this can be regarded as the function $s_\lambda(xy)$, where $x_1, \dots x_m$ are eigenvalues of an element of $SU(m)$, and $y_1, \dots y_n$ those of an element of $SU(n)$, and xy denotes the set of all products $x_i y_j$. The products $s_\mu(x) s_\nu(y)$ of Schur functions over all μ and ν with $|\mu| = |\nu| = |\lambda|$ are the characters of the irreducible representations of $SU(m) \times SU(n)$. Hence we can write $s_\lambda(xy)$ in this basis, and obtain the content expansion:

$$s_\lambda(xy) = \sum_{\mu, \nu} g_{\lambda \mu \nu} s_\mu(x) s_\nu(y). \tag{16}$$

The relationship between representations of $SU(d)$ corresponding to Eq. (16) can be written

$$\mathcal{R}_\lambda \downarrow_{SU(m) \otimes SU(n)} \cong \bigoplus_{\mu, \nu} g_{\lambda \mu \nu} \mathcal{R}_\mu \otimes \mathcal{R}_\nu, \tag{17}$$

where the left hand side denotes the representation $\mathcal{R}_\lambda$ of $SU(mn)$ restricted to the subgroup $SU(m) \times SU(n)$. Note that the underlying subspace of $\mathcal{R}_\mu \otimes \mathcal{R}_\nu$ is not in general $\mathcal{V}_\mu \otimes \mathcal{V}_\nu$, but is embedded in $(\mathcal{U}_\mu \otimes \mathcal{V}_\mu) \otimes (\mathcal{U}_\nu \otimes \mathcal{V}_\nu)$ by some unitary action.

The integers $g_{\lambda \mu \nu}$ are sometimes called the *Kronecker coefficients*; this name alludes to another context they occur in, as we now explain.

Let $\chi_\lambda(\tau)$ denote the character of the representation $\mathcal{U}_\lambda$ of S_k on the conjugacy class τ. The conjugacy class of a permutation in S_k is determined by the lengths of the cycles in the permutation, so τ is a partition of k whose parts τ_i represent the lengths of those cycles. Let $p_r(x_1, \dots, x_d) = \sum_i x_i^r$ be the rth *power sum* and let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_q}$. Then the character of the representation of $SU(d) \times S_k$ given by Eqs. (1) and (2) takes

the value $p_\tau(x) = p_\tau(x_1, \ldots, x_d)$ at an element of $SU(d)$ with eigenvalues $x_1, \ldots, x_d$ and a permutation with class $\tau \vdash k$. In terms of characters, therefore, Eq. (4) becomes

$$p_\tau(x) = \sum_\lambda \chi_\lambda(\tau) s_\lambda(x). \tag{18}$$

Thus

$$p_\tau(x) p_\tau(y) = \sum_{\mu\nu} \chi_\mu(\tau) \chi_\nu(\tau) s_\mu(x) s_\nu(y), \tag{19}$$

but since $p_\tau(x) p_\tau(y) = p_\tau(xy)$, by Eq. (16) we have

$$p_\tau(x) p_\tau(y) = \sum_{\lambda\mu\nu} g_{\lambda\mu\nu} \chi_\lambda(\tau) s_\mu(x) s_\nu(y). \tag{20}$$

Comparing Eqs. (19) and (20), and noting that the products $s_\mu(x) s_\nu(y)$ are linearly independent, we have for each μ,

$$\chi_\mu(\tau) \chi_\nu(\tau) = \sum_\lambda g_{\lambda\mu\nu} \chi_\lambda(\tau), \tag{21}$$

with $|\lambda| = |\mu| = |\nu|$. This implies that the corresponding representation subspaces satisfy

$$\mathcal{U}_\mu \otimes \mathcal{U}_\nu \cong \bigoplus_\lambda g_{\lambda\mu\nu} \mathcal{U}_\lambda. \tag{22}$$

Thus the Kronecker coefficients $g_{\lambda\mu\nu}$ appear in the Clebsch-Gordan series for the symmetric group S_k, i.e. in the series that represents the Kronecker (or tensor) product of two irreducible representations of S_k as a sum of irreducible representations, where the latter are weighted by the number of times they occur. This is analogous to the Clebsch-Gordan series for $SU(d)$,

$$s_\mu(x) s_\nu(x) = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda(x), \tag{23}$$

where $|\lambda| = |\mu| + |\nu|$. The coefficients $c_{\mu\nu}^\lambda$ can be calculated using the famous Littlewood-Richardson rule [8], whereas finding an efficient computational rule for the $g_{\lambda\mu\nu}$ is a fundamental open problem (see e.g. [7]).

Equation (21) and orthogonality of characters implies

$$g_{\lambda\mu\nu} = \frac{1}{k!} \sum_{\tau \in S_k} \chi_\lambda(\tau) \chi_\mu(\tau) \chi_\nu(\tau), \tag{24}$$

which shows that the Kronecker coefficients are symmetric under interchange of the indices.

IV. Main Result

We now show that there is a close correspondence between the spectra of a density oper-
ator ρ^{AB} and its traces ρ^A and ρ^B and Young frames λ, μ, ν with positive Kronecker
coefficients. More precisely,

Theorem 1. *For every density operator ρ^{AB}, there is a sequence $(\lambda_j, \mu_j, \nu_j)$ of parti-
tions, labelled by natural numbers j, with $|\lambda_j| = |\mu_j| = |\nu_j|$, such that*

$$g_{\lambda_j \mu_j \nu_j} \neq 0 \quad \text{for all} \tag{25}$$

and

$$\lim_{j \to \infty} \bar{\lambda}_j = spec(\rho^{AB}), \tag{26}$$

$$\lim_{j \to \infty} \bar{\mu}_j = spec(\rho^A), \tag{27}$$

$$\lim_{j \to \infty} \bar{\nu}_j = spec(\rho^B). \tag{28}$$

Proof. Let $r^{AB} = spec(\rho^{AB}), r^A = spec(\rho^A), r^B = spec(\rho^B)$. Let P_λ^{AB} denote the
projector onto the Young subspace $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$ in system AB, and let P_μ^A, P_ν^B be the cor-
responding projectors onto Young subspaces in A and B, respectively. By Corollary 2,
for given $\epsilon > 0$, we can find a k_0 such that the following all hold for all $k \geq k_0$,

$$\text{tr } P_X (\rho^A)^{\otimes k} \geq 1 - \epsilon, \quad P_X := \sum_{\bar{\mu} \in \mathcal{B}_\epsilon (r^A)} P_\mu^A, \tag{29}$$

$$\text{tr } P_Y (\rho^B)^{\otimes k} \geq 1 - \epsilon, \quad P_Y := \sum_{\bar{\nu} \in \mathcal{B}_\epsilon (r^B)} P_\nu^B, \tag{30}$$

$$\text{tr } P_Z (\rho^{AB})^{\otimes k} \geq 1 - \epsilon, \quad P_Z := \sum_{\bar{\lambda} \in \mathcal{B}_\epsilon (r^{AB})} P_\lambda^{AB}. \tag{31}$$

Equations (29) and (30) can be combined to yield

$$\text{tr } (P_X \otimes P_Y)(\rho^{AB})^{\otimes k} \geq 1 - 2\epsilon. \tag{32}$$

This follows from

$$\text{tr } (P \otimes Q)\xi^{AB} \geq \text{tr } P\xi^A + \text{tr } Q\xi^B - 1, \tag{33}$$

which holds for all projectors P and Q and density operators ξ^{AB} since $\text{tr } [(1 - P) \otimes
(1 - Q)\xi^{AB}] \geq 0$.
    Because $(\rho^{AB})^{\otimes k}$ maps each Young frame into itself, writing $\sigma = (\rho^{AB})^{\otimes k}$, we have

$$\sum_{\lambda \vdash k} P_\lambda \sigma P_\lambda = \sigma. \tag{34}$$

Defining $P_{\bar{Z}} := 1 - P_Z$, Eqs. (32) and (34) imply

$$\text{tr } [(P_X \otimes P_Y)(P_Z \sigma P_Z + P_{\bar{Z}} \sigma P_{\bar{Z}})] \geq 1 - 2\epsilon. \tag{35}$$

We now insert tr $[(P_X \otimes P_Y) P_{\bar{Z}} \sigma P_{\bar{Z}}] \leq \epsilon$ (from Eq. (31)) and obtain

$$\text{tr } [(P_X \otimes P_Y) P_Z \sigma P_Z] \geq 1 - 3\epsilon. \tag{36}$$

Clearly, there must be at least one triple $\mu \in \mathcal{B}_\epsilon(r^A)$, $\nu \in \mathcal{B}_\epsilon(r^B)$ and $\lambda \in \mathcal{B}_\epsilon(r^{AB})$ with tr $[(P_\mu^A \otimes P_\nu^B) P_\lambda^{AB} \sigma P_\lambda^{AB}] \neq 0$. Thus $(P_\mu^A \otimes P_\nu^B) P_\lambda^{AB} \neq 0$ and by Eq. (17) this implies $g_{\lambda\mu\nu} \neq 0$. $\quad\square$

Next we consider some consequences of the above theorem. The von Neumann entropy, $S(\rho)$, of the operator ρ is defined by $S(\rho) = -\text{tr } (\rho \log(\rho)) = H(r)$, where r is the spectrum of ρ.

Proposition 1. *Von Neumann entropy is subadditive; i.e. for all ρ^{AB}, $S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B)$.*

Proof. The Clebsch-Gordan expansion for the symmetric group, Eq. (22), implies

$$g_{\lambda\mu\nu} \dim \mathcal{U}_\lambda \leq \dim \mathcal{U}_\mu \dim \mathcal{U}_\nu. \tag{37}$$

Theorem 1 tells us that, for every operator ρ^{AB}, there is a sequence of non-vanishing $g_{\lambda_j\mu_j\nu_j}$ with $\bar{\lambda}_j$, $\bar{\mu}_j$, $\bar{\nu}_j$ converging to the spectra of ρ^{AB}, ρ^A and ρ^B. Since the Kronecker coefficients are always non-negative integers, if $g_{\lambda_j\mu_j\nu_j} \neq 0$, we have

$$\dim \mathcal{U}_{\lambda_j} \leq \dim \mathcal{U}_{\mu_j} \dim \mathcal{U}_{\nu_j}. \tag{38}$$

This holds for all j, and in the limit of large j Stirling's approximation and inequality (8) imply that $\frac{1}{k} \log(\dim \mathcal{U}_{\lambda_j})$ tends to $S(\rho^{AB})$, where $k = |\lambda_j|$, and similarly for systems A and B. Thus we obtain subadditivity. $\quad\square$

Proposition 2. *The triangle inequality [1], $S(\rho^{AB}) \geq |S(\rho^A) - S(\rho^B)|$, holds for all ρ^{AB}.*

Proof. The symmetry of the coefficients implied by Eq. (24) tells us that in addition to Eq. (38) we also have the two equations obtained by cyclically permuting λ, μ, ν; e.g. $\dim \mathcal{U}_{\mu_j} \leq \dim \mathcal{U}_{\lambda_j} \dim \mathcal{U}_{\nu_j}$. The triangle inequality then follows by applying the reasoning in the proof of the preceding proposition. $\quad\square$

Note that our proof of the triangle inequality is very different in spirit from the conventional one that applies subadditivity to the purification of the state.

Finally, we show that the non-vanishing of a Kronecker coefficient implies a relationship between entropies.

Proposition 3. *Let $\lambda, \mu, \nu \vdash k$. If $g_{\lambda\mu\nu} \neq 0$, then $H(\bar{\lambda}) \leq H(\bar{\mu}) + H(\bar{\nu})$, where $H(\bar{\lambda}) = -\sum_i \bar{\lambda}_i \log(\bar{\lambda}_i)$ is the Shannon entropy of $\bar{\lambda}$.*

Proof. Kirillov has announced ([7], Theorem 2.11) that $g_{\lambda\mu\nu} \neq 0$ implies $g_{N\lambda\,N\mu\,N\nu} \neq 0$, for any integer N, where $N\lambda$ means the partition with lengths $N\lambda_i$. Since $\frac{1}{Nk} \log(\dim \mathcal{U}_{N\lambda})$ tends to $H(\bar{\lambda})$ for large N, inequality (38) implies the result we seek. $\quad\square$

V. Conclusions

The ideas we introduce here build on concepts that were much in vogue in the particle physics of the 1960s. The $SU(m) \times SU(n)$ content of a representation of $SU(mn)$ expresses the types of symmetry possible in nuclei and elementary particles, e.g. in the multiplet theory of Wigner and in the eight-fold way of Ne'eman and Gell-Mann [11, 9, 2]. The connection that we derive between these concepts and the spectra of quantum states is novel and leads to surprisingly simple proofs of subadditivity of the von Neumann entropy and the triangle inequality. There are many ways to generalise these ideas, and exploration of them is likely to be fruitful.

Note. After this work was completed (quant-ph/0409016) Klyachko announced some very interesting results (quant-ph/0409113). These included a theorem closely related to our theorem 2, and also a converse that states that, if $g_{m\lambda, m\mu, m\nu} \neq 0$ for some positive integer m, then there is a density operator with the triple of spectra λ, μ, ν.

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Communicated by M.B. Ruskai