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# Nontriviality of equations and explicit tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank at least $2m-2$

J.M. Landsberg [1]

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ABSTRACT

For even (resp. odd) $m$, I show the Young-flattening equations for border rank of tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of [7] are nontrivial up to border rank $2m-3$ (resp. $2m-5$) by writing down explicit tensors on which the equations do not vanish. Thus these tensors have border rank at least $2m-2$ (resp. $2m-4$). The result implies that there are nontrivial equations for border rank $2n^2-n$ that vanish on the matrix multiplication tensor for $n \times n$ matrices. I also study the border rank of the tensors of [1] and the equations of [4]. I show the tensors $T_{2^k} \in \mathbb{C}^k \otimes \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$ of [1], despite having rank equal to $2^{k+1}-1$, have border rank equal to $2^k$. I show the equations for border rank of [4] are trivial in the case of border rank $2m-1$ and determine their precise non-vanishing on the matrix multiplication tensor.



## 1. Results and context

Let $A$, $B$, $C$ be complex vector spaces of dimensions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. A tensor $T \in A \otimes B \otimes C$ is said to have *rank one* if $T = a \otimes b \otimes c$ for some $a \in A, b \in B, c \in C$. More generally the *rank* of a tensor $T \in A \otimes B \otimes C$ is the smallest $r$ such that $T$ may be written as the sum of $r$ rank one tensors. Let $\hat{\sigma}_r^0 \subset A \otimes B \otimes C$ denote the set of tensors of rank at most $r$. This set is not closed (under taking limits or in the Zariski topology) so let $\hat{\sigma}_r$ denote its closure (the closure is the same in the Euclidean or Zariski topology). The variety $\hat{\sigma}_r$ is familiar in algebraic geometry, it is the cone over the $r$-th secant variety of the Segre variety, but we won't need that in what follows. The rank and border rank of a tensor are measures of its complexity. While rank is natural to complexity theory, border rank is more natural from the perspective of geometry, as one can obtain lower bounds on border rank via polynomials. Let $\mathbf{R}(T)$, $\underline{\mathbf{R}}(T)$ respectively denote the rank and border rank of $T$.

By [2], the maximum rank of a tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is at most $2\lceil \frac{m^3}{3m-2} \rceil$, although it is not known if this actually occurs for all $m$. The maximum border rank of a tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is $\lceil \frac{m^3}{3m-2} \rceil$ for all

*E-mail address:* jml@math.tamu.edu.

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$m \neq 3$ and five when $m = 3$, see [10,9]. It is an important problem to find explicit tensors of high rank and border rank, and to develop tests that bound the rank and border rank from below. For the border rank, such tests are in the form of polynomials that vanish on $\hat{\sigma}_r$. For rank the study is more complicated. All coordinate free lower bounds for rank that I am aware of arise from first proving a lower bound on border rank, and then taking advantage of special structure of a particular tensor to show its rank is higher than its border rank.

Perhaps the most important tensor for this study is the matrix multiplication tensor, where one considers matrix multiplication

$$M_n : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$$

as a tensor $M_n \in \mathbb{C}^{n^2*} \otimes \mathbb{C}^{n^2*} \otimes \mathbb{C}^{n^2} = A \otimes B \otimes C$. In [7], G. Ottaviani and I proved the bound $\underline{\mathbf{R}}(M_n) \geq 2n^2 - n$ by finding polynomials that vanished on $\hat{\sigma}_{2n^2-n-1}$ and showing these polynomials did not vanish on $M_n$. At the same time we found additional polynomials that vanished on $\hat{\sigma}_{2n^2-k}$ for $k = n, \dots, 2$, but these polynomials also vanished on $M_n$. However *we did not know whether or not these additional polynomials were identically zero*. The motivation for this paper was to show these additional polynomials are in fact not identically zero. To do this I write down an explicit sequence of tensors on which the polynomials do not vanish, see Theorem 1.2. These are the first proven nontrivial polynomials for border rank in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ beyond $2m - \sqrt{m}$. Since matrix multiplication satisfies these polynomials, the result raises the intriguing possibility that the border rank of matrix multiplication could be far less than I had previously expected.

My first hope had been to use the tensors of [1], as they had been shown to have high rank, but it turns out, see Proposition 1.6 below, that they have low border rank, which makes them all the more interesting.

A referee for an earlier version of this paper wrote that [4] contains equations for $\hat{\sigma}_r$ in the range $m + 1 \leq r \leq 2m - 1$. This turned out to be erroneous – the author of [4] had only claimed the equations were *potentially* nontrivial in this range. Since the equations are presented indirectly, it was difficult to determine their non-triviality in general (see Section 6 for a discussion), but I do show:

**Proposition 1.1.** *Let* $\dim A = \mathbf{a}$, $\dim B = \dim C = m$. *Then B. Griesser's equations of [4] for* $\hat{\sigma}_r$ *have the following properties:*

(1) *They are trivial for* $r = 2m - 1$ *and all* $\mathbf{a}$.
(2) *They are trivial for* $r = 2m - 2$, $\mathbf{a} \leq \frac{m}{2} + 2$, *in particular* $\mathbf{a} = m = 4$.
(3) *Setting* $m = n^2$, *matrix multiplication* $M_n$ *fails to satisfy the equations for* $r \leq \frac{3}{2}n^2 - 1$ *when* $n$ *is even and* $r \leq \frac{3}{2}n^2 + \frac{n}{2} - 2$ *when* $n$ *is odd, and satisfies the equations for all larger* $r$.

I was unable to determine whether or not the equations are trivial for $r = 2m - 2$, $\mathbf{a} = m$ and $m > 4$. If they are nontrivial for even $m$, they would give equations beyond the equations of [7].

In [4] the equations are only shown to be nontrivial on matrix multiplication for $r \leq n\lceil \frac{3n}{2} \rceil - 2$ and their non-triviality in general was not examined. Note that the bound for $n$ odd that (3) gives is $\underline{\mathbf{R}}(M_n) \geq \frac{3}{2}n^2 + \frac{1}{2}n - 1$, which equals Lickteig's bound of [8] which held the "world record" for over twenty years.

The equations of [7] are special cases of equations obtained via *Young flattenings* defined in [6], which I now review. The classical flattenings (which date back at least to Macaulay and Sylvester) arise by viewing $T \in A \otimes B \otimes C$ as a linear map $T : B^* \to A \otimes C$, and taking the size $(r + 1)$ minors (i.e., the determinants of the $(r + 1) \times (r + 1)$-submatrices), which give equations for $r \leq \min(\mathbf{b}, \mathbf{ac})$. These do not give all the equations, and the idea behind Young flattenings is to pass from multi-linear algebra to linear algebra in more sophisticated ways. The particular Young flattening used in [7] may be described as follows:

Let $\Lambda^p A \subset A^{\otimes p}$ denote the skew-symmetric tensors. Let $Id_{\Lambda^p A} : \Lambda^p A \to \Lambda^p A$ denote the identity map, and consider, assuming $p \leq \lfloor \frac{\mathbf{a}}{2} \rfloor$, the map $Id_{\Lambda^p A} \otimes T : \Lambda^p A \otimes B^* \to \Lambda^p A \otimes A \otimes C$. Compose this map with the skew-symmetrization map to get a map

$$T_A^{\wedge p} : \Lambda^p A \otimes B^* \to \Lambda^{p+1} A \otimes C. \tag{1.1}$$

If $\mathbf{R}(T) = 1$, then the linear map $T_A^{\wedge p}$ has rank $\binom{\mathbf{a}-1}{p}$. More precisely, if $T = a \otimes b \otimes c$, then the image of $T_A^{\wedge p}$ is the image of $\Lambda^p A \otimes a \otimes c$ under the skew-symmetrization map $\Lambda^p A \otimes A \otimes C \to \Lambda^{p+1} A \otimes C$. Thus if $\underline{\mathbf{R}}(T) \leq r$, then the size $\binom{\mathbf{a}-1}{p} r + 1$ minors of $T_A^{\wedge p}$ will be zero. These minors are the equations used in [7] to bound the border rank of matrix multiplication.

Now let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$, so when dealing with matrix multiplication, $m = n^2$. In the case $m$ is odd, the Young flattenings (potentially) give the best lower bounds by writing $m = 2p + 1$, so one potentially gets the bound $\underline{\mathbf{R}}(T) \geq \lceil \frac{2p+1}{p+1} m \rceil = 2m - 1$. When $m$ is even, one gets the best potential lower bound by writing $m = 2p + 2$, taking a $2p + 1$ dimensional subspace of $A$ and restricting the map to get the potential bound $\underline{\mathbf{R}}(T) \geq \lceil \frac{2p+1}{p+1} m \rceil = 2m - 2$.

In [7] it was shown these equations are nontrivial (i.e., do not vanish identically) when $m$ is a square up to $r = 2m - \sqrt{m}$ by showing they did not vanish on the matrix multiplication tensor $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$.

**Theorem 1.2.** *Let $p$ be a natural number. For all $m \geq 2p + 2$, the maximal minors of (1.1) give nontrivial equations for $\hat{\sigma}_r \subset \mathbb{C}^{2p+1} \otimes \mathbb{C}^{2p+2} \otimes \mathbb{C}^m$, the tensors of border rank at most $r$ in $\mathbb{C}^{2p+1} \otimes \mathbb{C}^{2p+2} \otimes \mathbb{C}^m$, up to $r = 4p + 1$. More generally, for all $k \geq 1$ and all $m \geq k(p + 1)$, they give nontrivial equations for $\hat{\sigma}_r \subset \mathbb{C}^{2p+1} \otimes \mathbb{C}^{k(p+1)} \otimes \mathbb{C}^m$ up to $r = k(2p + 1) - 1$.*

*For $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, this implies that when $m$ is even (resp. odd), the equations are nontrivial up to $r = 2m - 3$ (resp. $r = 2m - 5$).*

Thus the equations may be used to show that $\underline{\mathbf{R}}(T) \geq 2m - 2$ when $m$ is even (the best potential bound from these Young flattenings) and $\underline{\mathbf{R}}(T) \geq 2m - 4$ when $m$ is odd. I show the best possible bound may be attained also in the case when $m = 5$:

**Proposition 1.3.** *When $m = 5$, the determinant of (1.1) gives non-trivial equations for tensors of border rank 8 in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$.*

Regarding matrix multiplication, Theorem 1.2 implies the following corollary:

**Corollary 1.4.** *Let $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$ denote the matrix multiplication operator. Then $M_n$ satisfies nontrivial equations for the variety of tensors of border rank $2n^2 - n$.*

In [1] (also see [11]), setting $m = 2^k$, they give an explicit sequence of tensors $T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of rank $2m - 1$, see (4.1), and explicit tensors $T'_{m+1} \in \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^m$ of rank $3(m + 1) - k - 4$, see Section 5. Their tensors may be defined over an arbitrary field.

**Proposition 1.5.** *Let $m = 2^k$. The tensors $T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of [1] (see (4.1)) have border rank $m$, i.e., $\underline{\mathbf{R}}(T_m) = m < \mathbf{R}(T_m) = 2m - 1$.*

**Proposition 1.6.** *Let $m = 2^k$. The tensors $T'_{m+1} \in \mathbb{C}^m \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1}$ of [1] (see Section 5) satisfy $m + 2 \leq \underline{\mathbf{R}}(T'_{m+1}) \leq 2(m + 1) - 2 - k < \mathbf{R}(T'_{m+1}) = 3(m + 1) - 4 - k$.*

I expect the actual border rank to be close to the lower bound as many of the Young flattening equations vanish, even in the $p = 1$ case.

## 2. Proof of Theorem 1.2

Let $\dim A = 2p+1$. Note that if $T_j \in A\otimes B_j\otimes C_j$ for $j=1,\dots,k$ are such that $(T_j)_A^{\wedge p}$ is injective for each $j$, then taking $T = T_1+\cdots+T_k \in A\otimes(B_1\oplus\cdots\oplus B_k)\otimes(C_1\oplus\cdots\oplus C_k) =: A\otimes B\otimes C$, then $T_A^{\wedge p}$ will also be injective. One way to see this is to consider the surjectivity of the transpose maps $\Lambda^{p+1}A^*\otimes(C_1\oplus\cdots\oplus C_k)^*\to \Lambda^p A^*\otimes(B_1\oplus\cdots\oplus B_k)$. Then $\Lambda^{p+1}A^*\otimes C_j^*$ will surject onto $\Lambda^pA^*\otimes B_j$ for $j=1,\dots,k$, so the total map is surjective as well.

In [7] we showed that when $\dim A = 2p+1$, $\dim B=\dim C = p+1$, (1.1) is injective for the canonical tensor corresponding to multiplication of polynomials. That is, one takes $A = S^{2p}\mathbb{C}^2$, $B=C=S^p\mathbb{C}^{2*}$, and $T: B^*\times C^*\to A$ is just multiplication of polynomials. (In bases, $T(A^*)$ is just the space of $(p+1)\times(p+1)$ Toeplitz matrices.) Now just take $T$ to be the sum of $k$ such tensors, each with the same $A$, but different $B$, $C$ to prove the first assertion.

The second statement follows from the first by taking subspaces. When $m$ is odd one must consider $\mathbb{C}^{2p+1}\otimes\mathbb{C}^{2p+2}\otimes\mathbb{C}^{2p+2}\subset\mathbb{C}^{2p+3}\otimes\mathbb{C}^{2p+3}\otimes\mathbb{C}^{2p+3}$, which is why one does not get the optimal bound.

## 3. Proof of Proposition 1.3

I begin more generally by writing down tensors I expect to have nonzero determinant for (1.1) for all $m$ when $m$ is odd, and then show this is indeed the case when $m=5$. Let $a_{-p},\dots,a_p$ be a basis of $A=\mathbb{C}^{2p+1}$, $b_1,\dots,b_m$ a basis of $B=\mathbb{C}^m$, and $c_1,\dots,c_m$ a basis of $C=\mathbb{C}^m$.

Let $\lambda_{i,u}$ be numbers satisfying certain open conditions. (They may be chosen to be e.g., $\lambda_{i,u} = 2^{2^{i+u}}+2^u$.) Consider

$$T_{m,p}(\lambda) := \sum_{j=-p}^{-1} a_j\otimes\left(\sum_{\alpha=1}^{m-p+j-1}\lambda_{-j,\alpha}b_{j+p+1+\alpha}\otimes c_\alpha\right)+\sum_{j=0}^{p}a_j\otimes\left(\sum_{\beta=1}^{m-j}b_\beta\otimes c_{j+\beta}\right) \tag{3.1}$$

When $m=2p+1$, write $T_m(\lambda) = T_{2p+1,p}(\lambda)$.

For example, in matrices, when $p=1$ and $m=3$

$$T_3(\lambda) = a_{-1}\otimes\begin{pmatrix}0&0&0\\ \lambda_{1,1}&0&0\\ 0&\lambda_{1,2}&0\end{pmatrix}+a_0\otimes\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+a_1\otimes\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

Write $T=\sum a_j\otimes X_j$, where $X_j\in B\otimes C$, and use $X_0=\sum_{\alpha=1}^m b_\alpha\otimes c_\alpha : B^*\to C$ to identify $C$ with $B^*$, so $X_0$ becomes the identity matrix in $\mathfrak{gl}(B)$, the space of endomorphisms of $B$. Let $([X_i,X_j])$, $i,j\in\{-p,\dots,-1,1,\dots,p\}$ denote the $2mp\times 2mp$ block matrix, whose $(i,j)$-th block is the commutator $[X_i,X_j]=X_iX_j-X_jX_i$. By [5, p. 4], (1.1) is injective when $m=5$ if and only if the size $20\times 20$ matrix whose blocks are $[X_i,X_j]$ has nonzero determinant,

$$\det\bigl([X_i,X_j]\bigr)\neq 0. \tag{3.2}$$

**Remark 3.1.** Despite the simplicity of Eq. (3.2), I do not know how to prove it is related to border rank other than by expressing (1.1) in coordinates, making a choice of $a_0$, and applying elementary identities regarding determinants. It would be desirable to have a direct explanation.

We have $p=2$. Order the rows and columns by 2, 1, −1, −2, and then group the matrix $([X_i,X_j])$ into four blocks of equal size. Then the two diagonal blocks are zero, and the lower left and upper right blocks have the same determinant (up to sign). The lower left block of $([X_i,X_j])$ consists of four smaller blocks which are:

$$[X_2, X_{-1}] = \begin{pmatrix} 0 & \lambda_{1,2} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{1,3} - \lambda_{1,1} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,2} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{1,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[X_2, X_{-2}] = \begin{pmatrix} \lambda_{2,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2,2} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{2,3} - \lambda_{2,1} & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{2,2} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{2,3} \end{pmatrix}$$

$$[X_1, X_{-1}] = \begin{pmatrix} \lambda_{1,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{1,2} - \lambda_{1,1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{1,3} - \lambda_{1,2} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,3} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{1,4} \end{pmatrix}$$

$$[X_1, X_{-2}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_{2,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2,2} - \lambda_{2,1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{2,3} - \lambda_{2,2} & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{2,31} & 0 \end{pmatrix}.$$

The determinant of the lower left block is thus

$$(\lambda_{2,1}) \det \begin{pmatrix} \lambda_{1,2} & \lambda_{1,1} \\ \lambda_{2,2} & \lambda_{2,1} \end{pmatrix} \det \begin{pmatrix} \lambda_{1,3} - \lambda_{1,1} & \lambda_{1,2} - \lambda_{1,1} \\ \lambda_{2,3} - \lambda_{2,1} & \lambda_{2,2} - \lambda_{2,1} \end{pmatrix}$$
$$\cdot \det \begin{pmatrix} \lambda_{1,4} - \lambda_{1,2} & -\lambda_{2,2} \\ \lambda_{1,3} - \lambda_{1,2} & \lambda_{2,3} - \lambda_{2,2} \end{pmatrix} \det \begin{pmatrix} -\lambda_{1,4} & -\lambda_{2,3} \\ \lambda_{1,4} - \lambda_{1,3} & -\lambda_{2,1} \end{pmatrix} (-\lambda_{1,4}),$$

which is nonzero for a general choice of $\lambda_{i,u}$.

## 4. The tensors $T_m$ of [1]

I restrict to the case $m = 2^k$ because the other cases are similar only padded with zeros. In [1] they define tensors $T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A \otimes B \otimes C$ by

$$T_m := a_0 \otimes \left( \sum_{\beta=1}^{m} b_\beta \otimes c_\beta \right) + \sum_{j=1}^{k} a_j \otimes \left( \sum_{\alpha=1}^{2^{j-1}} b_\alpha \otimes c_{m-2^{\alpha-1}+1} \right) \tag{4.1}$$

Here I have changed the indices slightly from [1].

For example, when $k = 3$, in matrices, this is:

$$T_8(A^*) = \begin{pmatrix} a_0 &&&&&&& \\ & a_0 &&&&&& \\ && a_0 &&&&& \\ &&& a_0 &&&& \\ a_3 &&&& a_0 &&& \\ & a_3 &&&& a_0 && \\ a_2 && a_3 &&&& a_0 & \\ a_1 & a_2 && a_3 &&&& a_0 \end{pmatrix}.$$

If we reorder the basis of $B$ and write the tensor as

$$T_m = a_0 \otimes \left( \sum_{\beta=1}^{m} \tilde{b}_{m-\beta} \otimes c_\beta \right) + \sum_{j=1}^{k} a_j \otimes \left( \sum_{\alpha=1}^{2^{j-1}} \tilde{b}_{m-\alpha} \otimes c_{m-2^{\alpha-1}+1} \right) \tag{4.2}$$

We see this is a specialization of the multiplication tensor in $\mathbb{C}[X]/(X^m)$ whose border rank is $m$ (see, e.g., [3, Example 15.20]). Since $T_m : \mathbb{C}^{m*} \to \mathbb{C}^{k+1} \otimes \mathbb{C}^m$ is injective, we have the lower bound of $m$ as well. This proves Proposition 1.5.

## 5. The tensors $T'_{m+1}$ of [1]

In [1], they also define tensors $T'_m \in \mathbb{C}^{m+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^{m+1}$ by enlarging the matrices of $T_m$ to have size $m \times (m+1)$ and adding elements to the last column. For example, when $k = 3$ (so $m = 8$), one gets the $8 \times 9$ matrix

$$T'_9(A^*) := \begin{pmatrix} a_0 & & & & & & & & a_4 \\ & a_0 & & & & & & & a_5 \\ & & a_0 & & & & & & a_6 \\ & & & a_0 & & & & & a_7 \\ a_3 & & & & a_0 & & & & a_8 \\ & a_3 & & & & a_0 & & & \\ a_2 & & a_3 & & & & a_0 & & \\ a_1 & a_2 & & a_3 & & & & a_0 & \end{pmatrix}$$

which they express as a tensor in $\mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1}$ by adding zeros. These tensors have rank close to $3m$, to be precise, they show [1, Corollary 5.7] $\mathbf{R}(T'_m) = 3m - 2H(m-1) - \lfloor \log_2(m-1) \rfloor - 2$, where $H(m)$ is the number of 1's in the binary expansion of $m$, so the rank is best if $m - 1 = 2^k$, in which case $\mathbf{R}(T'_{2^k+1}) = 3(2^k + 1) - 4 - k$. The border rank is smaller. Write $T'_{m+1} = T_m + T''_m$ where $T''_m = (a_{k+1} \otimes b_1 + a_{k+2} \otimes b_2 + \cdots + a_m \otimes b_{m-k}) \otimes c_{m+1}$. Thus $\underline{\mathbf{R}}(T'_{m+1}) \leq \underline{\mathbf{R}}(T_m) + \underline{\mathbf{R}}(T''_m) = m + m - k$. One obtains the lower bound of $m + 2$ for the border rank (as opposed to the trivial $m + 1$) because the map $T_A^{\wedge 1}$ has a kernel of size $2^{k-1} = \frac{m-1}{2}$. Since this kernel is still quite large, I expect the actual border rank to be close to the lower bound.

## 6. The equations of [4]

Given $T = \sum_{j=0}^{\mathbf{a}-1} a_j \otimes X_j$ with $a_j$ a basis of $A$, $X_j \in B \otimes C$, $\dim A = \mathbf{a}$ and $\dim B = \dim C = m$, assume $X_0$ is of full rank and use it to identify $C$ with $B^*$. The equations in [4] are stated as: if the border rank of $T$ is at most $r$, with $m+1 \leq r \leq 2m-1$, then the space of endomorphisms $\langle [X_1, X_2], \ldots, [X_1, X_{\mathbf{a}-1}] \rangle \subset \mathfrak{sl}(B)$ is such that there exists $E \in G(2m-r, B)$, with $\dim(\langle [X_1, X_2], \ldots, [X_1, X_{\mathbf{a}-1}] \rangle(E)) \leq r - m$. Here $\langle \ldots \rangle$ denotes the linear span and $G(k, B)$ the Grassmannian of $k$ planes in $B$. Compared with Eq. (3.2), here one is just examining the last block column of the matrix appearing in (3.2), but one is extracting apparently more refined information from it.

Assuming $T$ is sufficiently generic, we may choose $X_1$ to be diagonal with distinct entries on the diagonal (a general element of $\mathfrak{sl}(B)$, the space of traceless endomorphisms, is diagonalizable with distinct eigenvalues), and this is a generic choice of $X_1$. Let $\mathfrak{sl}(B)_R$ denote the matrices with zero on the diagonal (the sum of the root spaces). Then $ad(X_1) : \mathfrak{sl}(B)_R \to \mathfrak{sl}(B)_R$, given by $Y \mapsto [X, Y]$, is a linear isomorphism, and $ad(X_1)$ kills the diagonal matrices. Write $U_j = [X_1, X_j]$, so the $U_j$ will be matrices with zero on the diagonal, and by picking $T$ generically we can have any such matrices, and this is the most general choice of $T$ possible, so if the equations vanish for a generic choice of $U_j$, they vanish identically.

**Proof of Proposition 1.1.** Proof of (1): In the case $r = 2m - 1$, so $r - m = m - 1$ and $\mathbf{a} \leq m + 1$ the equations are trivial as we only have $\mathbf{a} - 2 \leq m - 1$ linear maps. When $\mathbf{a} \geq m + 2$ a naïve dimension count makes it possible for the equations to be non-trivial, the equations are that $\dim\langle U_2 v, \dots, U_{\mathbf{a}-1} v\rangle \leq m - 1$. However, with our normalizations of $X_0 = Id$ and $X_1$ diagonal with distinct entries on the diagonal, taking $v = (1, 0, \dots, 0)^T$ (the superscript $T$ denotes transpose), the $U_j v$ will be contained in the hyperplane of vectors with their first entry zero. Since we only made genericity assumptions, we conclude.

Proof of (2): In the case $r = 2m-2$, the equations will be nontrivial if and only if there exist $U_2, \dots, U_{\mathbf{a}-1} \in \mathfrak{sl}(B)_R$ such that for all linearly independent $v$, $w$

$$\dim\langle U_2 v, \dots, U_{\mathbf{a}-1} v, U_2 w, \dots, U_{\mathbf{a}-1} w\rangle \geq m - 1.$$

The map ${U_2}^{-1} U_3$ will in general have $m$ linearly independent eigenvectors. Take $v$, $w$ to be two such, then $\langle U_2 v\rangle = \langle U_3 v\rangle$ and $\langle U_2 w\rangle = \langle U_3 w\rangle$, and the span will have dimension at most $2(\mathbf{a} - 2) - 2 \leq m - 2$, and we conclude.

**Remark 6.1.** I expect the equations are non-trivial for $m \geq 5$.

Proof of (3): Consider matrix multiplication $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} = A \otimes B \otimes C$. With a judicious choice of bases, $M_n(A^*)$ is block diagonal

$$\begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \tag{6.1}$$

where $x = (x^i_j)$ is $n \times n$. In particular, the image is closed under brackets. Choose $X_0$ so it is the identity. We may not have $X_1$ diagonal with distinct entries on the diagonal, the best we can do is for $X_1$ to be block diagonal with each block having the same $n$ distinct entries. For a subspace $E$ of dimension $2m - r = dn + e$ (recall $m = n^2$) with $0 \leq e \leq n - 1$, the image of a generic choice of $[X_1, X_2], \dots, [X_1, X_{n^2-1}]$ applied to $E$ is of dimension at least $(d + 1)n$ if $e \geq 2$, at least $(d + 1)n - 1$ if $e = 1$ and $dn$ if $e = 0$, and equality will hold if we choose $E$ to be, e.g., the span of the first $2m - r$ basis vectors of $B$. (This is because the $[X_1, X_j]$ will span the entries of type (6.1) with zeros on the diagonal.) If $n$ is even, taking $2m - r = \frac{n^2}{2} + 1$, so $r = \frac{3n^2}{2} - 1$, the image occupies a space of dimension $\frac{n^2}{2} + n - 1 > \frac{n^2}{2} - 1 = r - m$. If one takes $2m - r = \frac{n^2}{2}$, so $r = \frac{3n^2}{2}$, the image occupies a space of dimension $\frac{n^2}{2} = r - m$, showing Griesser's equations cannot do better for $n$ even. If $n$ is odd, taking $2m - r = \frac{n^2}{2} - \frac{n}{2} + 2$, so $r = \frac{3n^2}{2} + \frac{n}{2} - 2$, the image will have dimension $\frac{n^2}{2} + \frac{n}{2} > r - m = \frac{n^2}{2} + \frac{n}{2} - 1$, and taking $2m - r = \frac{n^2}{2} - \frac{n}{2} + 1$ the image can have dimension $\frac{n^2}{2} - \frac{n}{2} + (n - 1) = r - m$, so the equations vanish for this and all larger $r$. Thus Griesser's equations for $n$ odd give Lickteig's bound $\underline{\mathbf{R}}(M_n) \geq \frac{3n^2}{2} + \frac{n}{2} - 1$. $\square$

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