# Research interets

My research interests lie in the intersection of algebraic geometry, representation theory, differential geometry and singularity theory. More precisely I am interested in problems where those different fields of mathematics may provide different perspectives.

## My Phd Thesis

My Phd thesis focuses on two different problems which are two sides of the same question:

### Problems

• The first problem concerns hyperplane sections of smooth projective varieties. Let $X^n \subset \mathbb{P}(V)$ ($V = \mathbb{C}^{n+a+1}$) be a smooth complex projective variety and $H$ a hyperplane. Generically a hyperplane section of $X \subset \mathbb{P}(V)$ defines a smooth subvariety of $X$. Suppose $H$ is not generic, i.e. $H$ is tangent to some point $x \in X$, and suppose moreover that $x$, the point of tangency, is an isolated singular point of $X \cap H$. Then the restriction to $X$ of the linear form defining $H$ gives locally a singular hypersurface with isolated singularity. We denote that singular hypersurface by $(X \cap H, x)$.

**Problem 1:** How does the geometry of $X$ determine the possible singular types, in the sense of Arnold's classification, of $(X \cap H, x)$?

By Lefschetz's theorems the study of singular hyperplane sections of the variety $X \subset \mathbb{P}(V)$ provides information on the topology of $X$. For instance Fyodor Zak has proven (see [Zak 1973]) that the existence of a hyperplane section $(X \cap H, x)$ with isolated singular point $x$ which is not a Morse singularity, implies that the group of vanishing cycles of $X$ is not trivial.

• The second problem deals with dual varieties. The dual variety of a projective variety $X^n \subset \mathbb{P}(V)$ is the closure, in the dual projective space, of the set of tangent hyperplanes:

$$X^* := \overline{\{H \in \mathbb{P}(V^*),\ \exists\, x \in X_{smooth},\ \tilde{T}_x X \subset H\}} \subset \mathbb{P}(V^*)$$

Typically, when $X$ is smooth, the dual variety is a hypersurface of high degree. For example that is always true when $X$ is a complete intersection (other than the quadric which is codegree 2). Therefore two natural questions come up:

- What are the smooth varieties $X$ such that $X^*$ is not a hypersurface?
- What are the smooth varieties $X$ such that $\deg(X^*)$ is small?

There exist classification theorems about those two questions. Lawrence Ein classified smooth projective varieties whose dual varieties are the most defective ([Ein 1986]) under the assumption that the codimension of $X$ is not too small ($a \geq \frac{n}{2}$). Fyodor Zak provided a classification theorem for smooth projective varieties whose dual varieties are of degree 3 ([Zak 1993]).

Another typical property of the dual variety of a smooth projective variety is the one of being very singular. If $X$ is smooth and is not a hypersurface, $X^*$ can not be a smooth hypersurface. Also the dual variety of a complete intersection, other that the smooth quadric, is always singular in codimension 1.

That behaviour adds the following natural question about dual varieties:

**Problem 2:** What are the smooth varieties $X$ such that $X^*$ is normal?

**Connecting problems** 1 & 2

The condition on a hyperplane $H$ to be generic in the dual projective space is equivalent to require $H \notin X^* \subset \mathbb{P}(V^*)$. As we suppose $X$ to be smooth, then $H \notin X^*$ is equivalent to the condition $X \cap H$ is smooth. It is also well known that $X^*$ is a hypersurface and $H$ is a smooth point of $X^*$, is equivalent to $X \cap H$ has an unique singular point of type Morse singularity ($A_1$ in Arnold's classification).

Therefore one sees the connection between the two previous problems. The existence of singular hyperplane sections of a given type (other than an unique $A_1$ singular point) is related to the existence and dimension of some components of the singular locus of $X^*$.

**Main results of the thesis**

**Chapter one**

In the first chapter my purpose is to describe the singular locus of a special class of dual varieties coming from representation theory. Doing so I propose a new approch on the beautiful correspondance between simple singularities and simple Lie algebras.

Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $X_G \subset \mathbb{P}(\mathfrak{g})$ be the unique closed orbit for the projectived adjoint action of $G$ on $\mathbb{P}(\mathfrak{g})$. The variety $X_G \subset \mathbb{P}(\mathfrak{g})$ is called the adjoint variety of $G$. I propose to describe $X_G^* \subset \mathbb{P}(\mathfrak{g}^*)$ and its singular locus.

If $G$ is the Lie group $SL_{n+1}(\mathbb{C})$, i.e. $G$ is of type $A_n$, the adjoint variety is $(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}) \cap H_{\text{trace}=0} \subset \mathbb{P}(\mathfrak{sl}_{n+1})$. In other words in the $A_n$ case the adjoint variety identifies to the projectivization of the rank one traceless matrices. A computation shows that with that description, the dual variety $X_{A_n}^* \subset \mathbb{P}(\mathfrak{sl}_{n+1}^*)$ may be identify to the projectivization of the traceless matrices with reapeted eignevalue. Therefore one can choose for an equation of the dual variety the discriminant of the charactersitic polynomial. In others words, a matrix $A$ belongs to the dual if and only if

$$\Delta(t^{n+1} + P_2(A)t^{n-1} + \cdots + P_{n+1}(A)) = 0.$$

The $P_i$'s are the $SL_{n+1}$-invariant polynomials on $\mathbb{C}[\mathfrak{sl}_{n+1}]$, i.e. the sum of the principal $i \times i$ minors. The first polynomial $P_1$ corresponds to the trace and thus vanishes.

After identification $\mathfrak{sl}_{n+1} \simeq \mathfrak{sl}_{n+1}^*$ one can consider the map,

$$\Phi : \mathfrak{sl}_{n+1} \longmapsto \mathbb{C}^n$$

$$A \longmapsto (P_2(A), \ldots, P_{n+1}(A))$$

We denote by $\hat{X}_{A_n}^*$ the cone over the dual variety. Then $\Phi(\hat{X}_{A_n}^*)$ is the hypersurface defined by

$$\Delta(t^{n+1} + \lambda_1 t^{n-1} + \cdots + \lambda_n) = 0.$$

That hypersurface is the discriminant of the miniversal deformation of a simple singularity of type $A_n$. My first theorem is a generalization of this example:

Let us identify the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ via the Killing form. By Chevalley's theorem ones knows the ring of $G$-invariant polynomials over $\mathfrak{g}$ is finetely generated, i.e. $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \ldots P_n]$. The polynomials $P_i$ are not uniquely determined but their degrees are. We suppose those generators to be ordered for the degree, i.e. $\deg(P_i) \leq \deg(P_j)$ when $i \leq j$. The $P_i$'s allow us to define a quotient map which has been studied by Kostant ([Kos 1963]):

$$\Phi : \mathfrak{g} \longmapsto \mathfrak{g}//G \simeq \mathbb{C}^n$$
$$\Phi(x) = (P_1(x), \ldots, P_n(x)).$$

Finally let us denote the Dynkin diagram of our Lie algebra $\mathfrak{g}$ by $\Gamma$ and by $\Gamma^*$ the diagram of the long roots of $\Gamma$. We write $l$ for the number of long roots of $\Gamma$ and $\Delta_{\Gamma^*}$ the zero locus of a discriminant of simple singularity type $\Gamma^*$.

**Theorem (1.2.2 page 12).** *Let $\hat{X}_G^* \subset \mathfrak{g}^* \simeq \mathfrak{g}$ be the cone over the dual of the adjoint variety. Let $L$ be a linear subspace of $\mathfrak{g}//G \simeq \mathbb{C}^n$ of dimension $l$ given by $y_1 = \cdots = y_{n-l} = 0$, where $y_i$ are the linear coordinates on $\mathbb{C}^n$. Then $\Phi(X_G^*) \cap L = \Delta_{\Gamma^*}$.*

That theorem is a consequence of a theorem of Brieskorn for the type $A-D-E$. The $B-C-F-G$ diagrams are treated by a case by case computation.

Our theorem is the dual version of a theorem of Knop ([Kn 1987]) who described the singularities of the hyperplane sections of $X_G$. However our proof does not refer to any of Knop's computations and moreover our theorem gives new approaches on Knop's result. For instance by our theorem one can explain why certain hyperplanes chosen without explanation by Knop provide interesting hyperplane sections.

### Chapter two

In the second chapter I investigate the link between singular hyperplane sections and singular locus of the dual from a more general point of view. I start with $X \subset \mathbb{P}(V)$ a smooth projective variety and I assume that $X^*$ is a hypersurface. In the spirit of the Plücker formula, relating plane curves and their dual curves, I decompose the singular locus of the dual, $\mathrm{Sing}(X^*)$, in two components. The first component denoted by $X^*_{cusp}$ will be the set of tangent hyperplanes such that the hyperplane section is singular but the singularity is not an ordinary quadric. The second component is denoted by $X^*_{node}$, it is the closure of the set of tangent hyperplanes with at least two points of tangency. The variety $X$ being smooth, one has $\mathrm{Sing}(X^*) = X^*_{cusp} \cup X^*_{node}$. Instead of working directly with the cusp and node components I propose to look at two geometric objects. Let us denote by $\tau(X)$ the tangential variety of $X$ and by $\sigma(X)$ the secant variety of $X$. Then one can show that $\tau(X)^* \subset X^*_{cusp}$ and $\sigma(X)^* \subset X^*_{node}$. I give criteria insuring $\tau(X)^*$ and $\sigma(X)^*$ are of maximal dimension. The conditions on dimension translate to conditions on hyperplane sections:

**Theorem (2.2.3 page 41).** *Let $X^n \subset \mathbb{P}(V)$ be a smooth projective variety and $x$ a general point of $X$. Let $|II_{X,x}|$ denote the system of quadrics defined by the second fundemental form and let $\tilde{T}_x^{(2)}X$ denote the second osculating space to $X$ at $x$. We assume $\sigma(X)$ is not degenerate (i.e. of dimension $2n+1$).*

*- If $\exists\ Q \in |II_{X,x}|$ of rank $n-1$, such that $ker(Q)$ is a generic direction of $T_xX$, then $\exists\ H \in \mathbb{P}(V^*)$ such that $(X \cap H, x)$ is a singularity of type $A_2$.*

*- If $(x, y) \in X \times X$ is a general pair of points and $\tilde{T}_xX \cap \tilde{T}_yX = \emptyset$, then $\exists\ H$ such that $(X \cap H, x)$ and $(X \cap H, y)$ are Morse singular points.*

### Chapter 3

In the last chapter I solve problem 2 for homogeneous varieties. By homogeneous varieties I mean the unique closed orbit given by the action of a semi-simple Lie group $G$ on $\mathbb{P}(V)$, where $V$ is an irreducible representation of $G$. It should be mentioned here that the question of the dimension of the dual variety of a homgeneous variety has been solved by Knop and Menzel ([K-M 1987]) and that there exist results on the degree of the dual for homgeneous varieties. For instance the case where $X = \mathbb{P}^{k_1} \times \dots \mathbb{P}^{k_r} \subset \mathbb{P}^{(k_1+1)\dots(k_r+1)-1}$, i.e. $X$ is a Segre embbeding of $r$-projective spaces with $k_1 \leq k_2 + \dots k_r$, was studied in detail by Gelfand, Kapranov and Zelevinsky. Under the condition $k_1 \leq k_2 + \dots k_r$ the dual is always a hypersurface. One calls hyperdeterminant the equation (unique up to scalar multiplication) which defines that hypersurface. Gelfand, Kapranov and Zelevinsky prove in [G-K-Z 1992] a formula to compute the degree of hyperdeterminants. Later on Weyman and Zelevinsky [W-Z 1996] studied the singular locus of hyperdeterminants and prove there exists only one normal hyperdeterminant. That normal hyperdeterminant is obtained if and only if $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. The following theorem generalizes to all homogeneous varieties whose dual is a hypersurface the result of Weyman and Zelevinsky on normality of hyperdeterminants.

**Theorem (3.2.5 page 71).** *Let $X = G/P \subset \mathbb{P}(V)$ be a rational homogenous projective variety such that its dual is a hypersurface. The dual variety $(G/P)^* \subset \mathbb{P}(V^*)$ is normal if and only if $X$ is one of the following varieties,*

-$\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ *(the smooth quadric).*

-$v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{(n+2)(n+1}{2}-1}$, $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{(n+1)^2-1}$, $G(2,2n) \subset \mathbb{P}^{\binom{2n}{2}-1}$, $\mathcal{E}_6 \subset \mathbb{P}^{26}$ *(a Scorza variety).*

-$G_\omega(3,6) \subset \mathbb{P}^{13}$, $G(3,6) \subset \mathbb{P}^{19}$, $\mathbb{S}_6 \subset \mathbb{P}^{31}$, $\mathcal{E}_7 \subset \mathbb{P}^{55}$, $\mathbb{P}^1 \times \mathbb{Q}^n \subset \mathbb{P}^{2(n+2)-1}$ *(a Legendrien homogeneous variety).*

-$X_{G_2} \subset \mathbb{P}(\mathfrak{g}_2)$ *(the adjoint variety for the simple Lie group $G_2$).*

To prove that theorem one computes, thanks to the results of chapter two, the dimension of $\sigma(G/P)^* \subset (G/P)^*$. The component $\sigma(G/P)^*$, being a subvariety of the singular locus of $(G/P)^*$, one deduces $(G/P)^*$ is not normal when $\sigma(G/P)^*$ has codimension one (Serre's criterion). The computation for the dimension of $\sigma(G/P)^*$ (pages 46 to 59) rules out alot of cases and we end up a finite list of homogeneous varieties whose dual may or may not be normal hypersurfaces. We achieve the proof by using a criterion due to Fyodor Zak [Zak 1989] combined with a computation on the second fundamental form to exclude some more varieties. The remaning varieties are the quadrics, the Scorza varieties, the Legendrien homogeneous varieties (all of them are known to have normal dual varieties) and the $G_2$-adjoint variety. That variety has not been so much studied in terms of its dual but with the theorem 1.2.2 of chapter one I can prove the dual of the $G_2$ adjoint variety is normal.

The next natural question is to solve the problem for all $(G/P)^*$. By Knop-Menzel's theorem there is a finite number of defective $(G/P)^*$.

**Theorem (3.4.1 page 76).** *Let $X = G/P \subset \mathbb{P}(V)$ be a rational homogeneous projective variety. The dual variety $X^*$ is normal if and only if $X$ is one of the following varieties,*

- *A variety of theorem 3.2.5.*
- $\mathbb{S}_5 \subset \mathbb{P}^{15}$.
- $\mathbb{P}^k \times \mathbb{P}^l \subset \mathbb{P}^{(k+1)(l+1)-1}$ $k > l$.
- $G(2, 2n+1) \subset \mathbb{P}^{n(2n+1)-1}$.
- $\mathbb{P}^m \times \mathbb{Q}^1 \subset \mathbb{P}^{3m+1}$, $m > 1$.

## Future research plans

Here is a list of questions which are possible prolongations of my work:

### Simple Lie algebras and simple singularities

Let us recall two constructions which connect simple singularities to simple Lie algebras . The first one is the Brieskorn-Grothendieck-Slodowy construction. Brieskorn's theorem ([Br 1970]) insures that one can construct a simple singularity of type $A-D-E$ by considering the nilpotent orbit in the projectivization of a Lie algebra of type $A-D-E$. Cutting the nilpotent orbit transversly to its singular suborbit provides a surface with an isolated singular point of the desired type. The second construction, due to Knop, deals with the adjoint variety of a Lie algebra of type $A-D-E$. In that construction the simple singularity of type $A-D-E$ is obtained by taking a specific hyperplane section of the adjoint variety.

The theorem 1.2.2 gives a new (dual) perspective on the $A-D-E$ correspondance. Moreover as mentioned before it explains some aspects of Knop's theorem. With some case by case computations I also can use theorem 1.2.2 to give a more direct proof of Knop's theorem.

Two questions remain important to me:

- Can I give a proof of Knop's theorem without a case by case computation ?

One encouraging step in that direction is that the equation defining the dual variety of adjoint varieties can be described uniformly for all simple Lie algebras.

- Can I link both Brieskorn and Knop's constructions ? In the proof of the theorem 1.2.2 I do not use anywhere the theorem of Knop but I do use a consequence of Brieskorn's theorem. It would be interesting to show that both constructions are equivalent.

### Understanding the varieties of theorem 3.2.5 in terms of series

Fyodor Zak asked me the following: can we understand the varieties involved in the theorem 3.2.5 in terms of series?

In my opinion it would be interesting to look at *homaloid polynomials* in order to answer that question. Chaput has proven in [Ch 2003] that the homaloid polynomials of degree three were exactely the defining equations of the dual varieties of the Scorza varieties. He also proposes as candidates for a classification of homaloid polynomials of degree four the equations of the dual varieties of the Legendrien homogeneous varietes. Thus proving Chaput's guess would be an encouraging starting point. Then the question of homoloid polynomial of degree 5 and 6 should be asked.

### More examples of complete decomposition of $Sing((G/P)^*)$

In [W-Z 1996], Weyman and Zelevinsky give a complete description of the singular locus of hyperdeterminants. Their paper shows that typically the singular locus of the hyperdeterminant has two components of codimension one (the cusp and the node components). The most pathological case is when $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is the only one case where the hyperdeterminant is normal. Then between that case and the typical behaviour, there exits a zoo of hyperdeterminants whose singular locus is of codimension one, but not necessary formed by the two principal components.

I expect the same to happen with all homogeneous varieties. More precisely it would be nice to prove some classifications of type:

($i$) there is a few $G/P$ which have normal dual variety (this is already proven by theorem 3.4.1)

($ii$) there exits a zoo of $G/P$ such that $(G/P)^*$ is not normal but its singular locus is not as expected and I would like to describe the singular locus of each member of the zoo.

($iii$) when $G/P$ is not a variety of ($i$) or ($ii$), then the singular locus of $(G/P)^*$ is the union of the dual of the tangential variety and the dual of the secant variety.

For instance if $G/P = v_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ I can prove that $G/P$ is in ($i$) if and only if $d = 2$ and otherwise $G/P$ is in ($iii$). I started computations with Grassmanian varieties which indicate some similarity with the case of hyperdeterminants. The goal in studing ($ii$) and ($iii$) would be to make connections with representation theory and orbit classifications.

## References


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