411 Honors homework due 3/18/19

Write a finite probability distribution as $\overline{p} = (p_1, \cdots, p_d)$, where $p_i \geq 0$ and $\sum_i p_i = 1$. The goal of this homework is to prove:

Theorem 0.1. *Let h be a function on finite probability distributions satisfying*

(1) *For each fixed d, h is continuous in the p_i.*

(2) *If $p_i = \frac{1}{d}$ for all i and we increase d, h is monotonically increasing.*

(3) *If we partition $\{1, \cdots, d\}$ into sets $A_1, \cdots, A_n$, and $\mathrm{Prob}(A_t) > 0$ for $t = 1, \cdots, n$, and set $\overline{q} = (\mathrm{Prob}(A_1), \cdots, \mathrm{Prob}(A_n))$, then $h(\overline{p}) = h(\overline{q}) + \sum_{t=1}^{n} \mathrm{Prob}(A_t) h(\overline{p}(\cdot|A_t))$ where $h(\overline{p}(\cdot|A_t))$ denotes the conditional probability distribution.*

Then up to a constant factor, h is the Shannon entropy $H(\overline{p}) = -\sum_i p_i \log(p_i)$.

(1) Set $A(d) = h(\frac{1}{d}, \cdots, \frac{1}{d})$ Show that $A(s^m) = mA(s)$. Hint: use property 3, writing $\{1, \cdots, s^m\} = \{s_{i,t} | 1 \leq i \leq s,\ 1 \leq t \leq s^{m-1}\}$ and set $A_t = \{s_{i,t} | 1 \leq i \leq s\}$.

(2) Show $A(s) = C \log(s)$ for some constant C.

(3) Now say we have a choice of D equally likely outcomes, which we break up as $D = \sum_{i=1}^{d} \delta_i$ Write $p_i = \frac{\delta_i}{D}$ and assume the δ_i are natural numbers. Show that $h(p_1, \cdots, p_d) = -C \sum_i p_i \log(p_i)$.

(4) Finish the proof using property 1.