(1) Let $X \subset \mathbb{P}V = \mathbb{P}^N$ be a projective variety, and let $y \in \mathbb{P}V$ be a point. Define the cone over $X$ with vertex $y$,

$$J(X, y) := \bigcup_{x \in X} \{x, y\}$$

where $\{x, y\}$ is the projective span of $x$ and $y$ (a $\mathbb{P}^1$ if $x \neq y$). Show that one needs the Zariski closure in the definition if and only if $y \in X$.

(2) Assume $y = [1,0,\ldots,0]$. Show that $P \in \mathcal{I}_{J(X,y)}$ if and only if $\frac{\partial^j P}{\partial x_1^j} \in \mathcal{I}_X$ for all $0 \leq j \leq \deg(P)$.

Remark 0.1. More generally for $X, Z \subset \mathbb{P}V$, one can define $J(X, Z)$, the join of $X$ and $Z$ to be

$$J(X, Z) := \bigcup_{x \in X, z \in Z} \{x, y\}.$$  

When $X = Z$, $J(X, X)$ is called the secant variety of $X$ and is denoted $\sigma(X) = \sigma_2(X)$. More generally, one defines $\sigma_r(X) := J(X, \sigma_{r-1}(X))$, the variety of secant $\mathbb{P}^{r-1}$'s to $X$. How would you find the ideals of these varieties given the ideals of $X$ and $Z$?